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Research Paper

VIBRATION AND BUCKLING OF COMPOSITE BEAMS WITH PARTIAL SHEAR INTERACTIONS USING HBT

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Free vibration and buckling analysis are presented for composite beams with partial using a finite element model developed by the authors based on a higher order beam theory (HBT). The proposed model takes into account the effect of partial shear interaction between the adjacent layers as well as transverse shear deformation of the beam. A third order variation of the axial displacement of the fibres over the beam depth is taken to have a parabolic variation of shear stress which is also made zero at the beam top and bottom surfaces. In the proposed FE model, there is no need of incorporating any shear correction factor and the model is free from shear locking problem. The proposed finite element model is validated by comparing the results with those available in literature. Many new results are presented as there is no published result on vibration and buckling of composite beams based on higher order beam theory.

Keywords: Composite beam, Partial shear interaction, Higher order beam theory, Finite element, Vibration, Buckling

INTRODUCTION

Composite beams are widely used in many engineering applications which utilise the best material properties of their components. A common use of these composite structures is the steel-concrete composite beam where concrete is utilized to resist the compression and steel takes the tension developed in it. The overall behaviour of these structures largely dependents on the type of connectors used to join the steel and concrete components where the connectors help to transfer the shear stress from one component to the other to have a composite action. A mechanical shear connector such as steel shear stud is quite common in steel concrete composite beams. A full composite action may be obtained theoretically by taking a very strong shear connector having infinite stiffness to eliminate any interfacial shear slip, i.e., a full shear interaction can be achieved with a rigid connection. However, in practice, a rigid

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connection is hardly realized due to the deformability of the shear connectors having finite stiffness. Thus a partial shear interaction is always observed due to interfacial slip between the two layers or components of a composite beam (Oehlers and Bradford, 1995). The problem has been identified long back and it has been studied by different investigators which show that the partial shear interaction has significant effect on the structural behaviour and it must be considered in the analysis of composite beams. The partial shear interaction has similar importance in other type of composite beams such as layered wooden beams, wood-concrete floor system and few other civil engineering problems. Moreover, some common modelling issues including partial shear interaction are also found in multilayered laminated composite structures made of fibre reinforced polymer composite materials which are becoming popular in aerospace, automotive, underwater and civil engineering problems. However, the present study will be focused on two layered composite beams having a single shear flexible interface such as steel-concrete composite beams or two layer wooden beams.

One of the earliest and most cited works on the partial interaction of composite beams is due to Newmark *et al.* (1951) and it is based on small deformation elastic analysis considering Euler-Bernoulli's beam theory for representing the deformation of beam layers. By this time a large number of investigations have been carried out by the various researchers on different aspects of composite beams and the literature is so vast that it is not feasible to give a full account of all these. However, some important studies relevant to the present research are mentioned as representative references (Goodman and Popov, 1968; Girhammar and Pan, 1993; Arzumi *et al*., 1981; Jasim, 1997; Salari *et al*., 1998; Ayub and Filippou, 2000; Wu *et al*., 2002; Faella C*et al*., 2002; Dall'Asta and Zona, 2002; Ranzi *et al*., 2004; Ranzi *et al*., 2006; and Schnabl *et al*., 2006). All these studies are based on Euler-Bernoulli's beam theory (EBT) which does not consider any effect of transverse shear deformation assuming the planes perpendicular to the beam axis before bending will remain plane and perpendicular to the curved beam axis after deformation. As the effect of transverse shear deformation is not small for beams having small span to depth ratio, low shear rigidity or continuous spans, the shear deformation has been incorporated in the analysis in some recent investigations (Berczynski and Wroblewski, 2005; Xu and Wu, 2007; and Schnabl *et al*., 2007) who used Timosheko's beam theory (TBT) to represent the deformation of the beam layers. A slightly different approach has been adopted by Ranzi and others (Ranzi and Zona, 2007; Ranzi, 2008; Ranzi *et al*., 2010; and Zona and Ranzi, 2011) to analyse steel concrete composite beams where the EBT is used to model the concrete slab while the steel girder is modelled with TBT.

In Timosheko's beam theory, it is assumed that any plane perpendicular to the beam axis before bending remains plane but not necessarily perpendicular to the beam axis after deformation. This helps to incorporate the effect of transverse shear deformation simply by taking the additional rotation of these planes as the shear strain which gives a uniform shear stress distribution over the beam depth whereas the actual variation of shear stress is parabolic and it becomes zero at the beam top and bottom surfaces. It causes warping of the beam section over its depth. In order to get a satisfactory result with this simplification made in TBT, the shear stiffness of the beam is modified with a factor known as shear correction factor which is dependent on the cross-sectional area of the beam. For a single layer homogeneous beam having a rectangular cross-section, the shear correction factor is 5/6 which can be evaluated by equating the strain energy in shear. This is simply used by some researchers for the analysis of composite beams also but the shear correction factor should have a different value depending on the geometry and material properties of the bounding layers and their interfacial slip. An accurate estimation of the shear correction factor will be quite complex. This may be obtained following the approach of Whitney (1973) who evaluated shear correction factors of multilayered composite laminates having rectangular cross-section considering perfect interface i.e. a full shear interaction is taken at the interfaces between the layers.

Moreover, an analysis based on TBT cannot satisfactorily predict various structural responses closer to the exact solutions. In order to overcome all these problems, a higher order beam theory (HBT) is used to model the deformation of the composite beam in this study. The HBT incorporates the warping of the beam section produced by shear deformation by taking a nonlinear variation of the axial displacement of the fibres over the beam depth. The higher order theories are often used for the analysis of multilayered laminated composite structure but these theories has not been applied to steel-concrete composite beams and similar composite beams. The present study has utilised the concept of the higher order shear deformation theory proposed by Reddy (1984) for multilayered composite laminates having full shear interaction at the interface between the layers. Reddy (1984) took third order variation of the in-plane displacements of the plies over the laminate thickness to have a quadratic/parabolic variation of shear strains/ stresses over the laminate thickness. The shear stresses become zero at the top and bottom surfaces of the laminate which is also satisfied in this theory (Reddy, 1984) and it helped to eliminate the additional unknowns used to express the cubic variation of the in-plane displacements over the laminate thickness. Moreover, Reddy's theory (Reddy, 1984) only retain usual nodal unknowns with physical interpretation which is not found in some other higher order theories (Kant, 1982) and this made it (Reddy, 1984) most elegant.

In this study, Reddy's theory (Reddy, 1984) is applied to both the components of material layers of the composite beams separately where the centroidal axis of the two components are taken as their reference axes. In this situation the shear stress free condition cannot be utilised at the bottom surface of the upper component and also at the top surface of the lower component which is practically the interface between these two components. The problem is overcome by taking axial displacement at the top and bottom surfaces of the interface as unknowns and it helped to get interfacial slip easily which is used with the interfacial shear stiffness to combine the two components of the composite beam. As the possibility of having vertical separation between the two layers is remote under static condition, the transverse displacement is taken to be same for both the layers. The higher order theory (Reddy, 1984) is simple and elegant as mentioned earlier but the finite element formulation of this theory demands C^1 continuity of the transverse displacement *w* as it involves second order derivative of *w* in the strain components. For a beam problem, the satisfaction of this continuity requirement is not that difficult as that found in plate/shell elements but a C^0 continuous finite element formulation is always attractive due to its computational elegance. This problem has drawn s significant attention of various researchers and a number of investigations are carried out for finding out a satisfactory solution of this problem. One of these techniques (Cook *et al.*, 2002) based on a penalty function approach is quite attractive which will be utilised in this study to develop a C^0 continuous composite beam element based on the quadratic isoparametric formulation. In order to make the element free from shear locking and stress oscillation problems the field consistent technique proposed by Vinayaka *et al.* (1996) is used which also helped to improve the convergence of the proposed model. The details of the element are presented in the formulation section.

Xu and Wu (2007) investigated the static, dynamic and buckling behaviour of composite beams with partial interactions by using TBT. The effect of rotary inertia in addition to shear deformation was included in the formulation to report the vibration frequencies of composite beams with partial interaction. Berczynski and Wrsblewski (2005) and Ranzi and Zona (2007) also analysed the vibration of steel concrete composite beams using TBT.

Xu and Wu (2007) presented a two dimensional analytical solution for the static analysis of simply supported composite beams with interlayer slip. Xu and Wu (2008) also presented the free vibration and buckling analysis of composite beams with interlayer slip in line with the previous two-dimensional theory. They presented results for a two layered wood-concrete beam for different boundary conditions considering only one value of interface spring stiffness.

Chakrabarti *et al.* (2012) recently proposed a new one dimensional finite element model for the static analysis of composite beams with partial shear interactions using a higher order beam theory (HBT). This one dimensional model gives results closer to the exact solutions and represents the variation of shear stresses across the depth in a better way. In this paper, the proposed one dimensional finite element model based on HBT is applied to solve the problems of free vibration and buckling of composite beams with partial interaction. The model is first validated by solving some bench mark problems of free vibration and buckling of composite beams having partial interactions and the convergence of the results is also tested. The results show an excellent performance of the beam finite element in predicting the free vibration frequencies and buckling loads of the composite beams. Some new problems are solved, which include different boundary conditions, cross-sectional geometry, and interfacial spring stiffness, and the results are presented for future references.

FORMULATION

The focus of the present study is to investigate the behaviour of composite beams having two material layers with a shear flexible interface as shown in Figure 1. As discussed in the previous section, a third order variation of the axial displacement of the fibres over the beam depth is taken and for the upper layer, it can be expressed as

$$u_c(x, y_c, z) = u_{c0}(x) - y_c \quad \theta_c(x) + y_c^{\ 2} \quad \alpha_c(x) + y_c^{\ 3} \quad \delta_c(x)$$

or

$$u_c = u_{c0} - y_c \quad \theta_c + y_c^{\ 2} \quad \alpha_c + y_c^{\ 3} \quad \delta_c \qquad \text{...(1)}$$

where u_c is the axial displacement at the reference axis of the upper layer passing through its centroid, θ_c is the bending rotation, and α_c and δ_c are the higher order terms. For the lower layer, this can be similarly expressed as

$$u_s = u_{s0} - \theta_s \quad y_s + \alpha_s \quad y_s^{\ 2} + \delta_s \quad y_s^{\ 3} \qquad \text{...(2)}$$

The transverse displacement is taken to be same for both layers as mentioned earlier and it can be expressed as

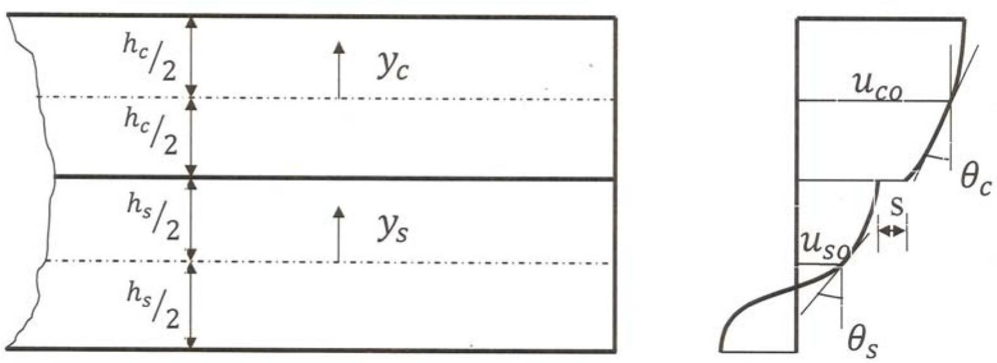
$$w_c(x, y_c, z) = w_s(x, y_s, z) = w(x) = w \qquad \text{...(3)}$$

The partial shear interaction of the two material layers is modelled with distributed shear springs connecting the two layers at their interface where the stiffness of these distributed elastic springs and the interfacial shear slip can be used to evaluate the interfacial shear stress. The interfacial slip can simply be obtained from the axial displacement at the bottom surface of the upper layer (u'_c) and that at the top surface of the lower layer (u'_s) as

$$s = u'_s - u'_s \qquad \text{...(4)}$$

As the higher order terms found in Equations (1) and (2) do not have any physical meaning, it will cause difficult to impose boundary conditions in an unknown problem. Thus these non-physical terms are eliminated with the help of shear stress free conditions at the top and bottom surfaces of the beam and taking u'_c and u'_s as independent unknowns. With the help of Equations (1) and (3), the shear stress at any point of the upper material layer may be expressed as

Figure 1: Two Layered Composite Beam with Shear Flexible Interface



$$\tau_c = G_c \gamma_c = G_c \left(\frac{\partial u_c}{\partial y_c} + \frac{\partial w}{\partial x} \right)$$
$$= G_c \left(-\theta_c + 2y_c \alpha_c + 3y_c^2 \delta_c + \frac{dw}{dx} \right) \quad ...(5)$$

where γ_c is the shear strain at that point and G_c is the shear modulus of the material. The shear stress free condition at the beam top surface can be obtained by substituting $y_c = h_c/2$ in the above equation as

$$-\theta_c + h_c \alpha_c + \frac{3}{4} h_c^2 \delta_c + \frac{dw}{dx} = 0 \quad ...(6)$$

The shear stress at any point of the lower layer can be obtained in a similar manner using Equations (2) and (3) as

$$\tau_s = G_s \gamma_s = G_s \left(\frac{\partial u_s}{\partial y_s} + \frac{\partial w}{\partial x} \right)$$
$$= G_s \left(-\theta_s + 2y_s \alpha_s + 3y_s^2 \delta_s + \frac{dw}{dx} \right) \quad ...(7)$$

Similarly the shear stress free condition at the bottom surface of the beam can be obtained by substituting $y_s = -h_s/2$ in the above equation as

$$-\theta_s - h_s \alpha_s + \frac{3}{4} h_s^2 \delta_s + \frac{dw}{dx} = 0 \quad ...(8)$$

Again can be obtained from Equation (1) by substituting $y_c = -h_c/2$ as

$$u'_c = u_{c0} + h_c \theta_c + \frac{h_c^2}{4} \alpha_c - \frac{h_c^3}{8} \delta_c \quad ...(9)$$

Similarly can be obtained from Equation (2) by substituting $y_s = -h_s/2$ as

$$u'_s = u_{s0} - h_s \theta_s + \frac{h_s^2}{4} \alpha_s + \frac{h_s^3}{8} \delta_s \quad ...(10)$$

Now the non-physical higher order terms can be expressed in terms of other usual terms with the help of Equation (6), Equation (8), Equation (9) and Equation (10) as follows.

$$\alpha_c = \frac{12}{5h_c^2}(u'_c - u_{c0}) - \frac{6}{5h_c}\theta_c - \frac{2}{5h_c}\left(\frac{dw}{dx} - \theta_c\right) \quad ...(11)$$

$$\alpha_s = \frac{12}{5h_s^2}(u'_s - u_{s0}) - \frac{6}{5h_s}\theta_s - \frac{2}{5h_s}\left(\frac{dw}{dx} - \theta_s\right) \quad ...(12)$$

$$\delta_c = -\frac{16}{5h_c^3}(u'_c - u_{c0}) + \frac{8}{5h_c^2}\theta_c - \frac{4}{5h_c^2}\left(\frac{dw}{dx} - \theta_c\right) \quad ...(13)$$

$$\delta_s = \frac{16}{5h_s^3}(u'_s - u_{s0}) + \frac{8}{5h_s^2}\theta_s - \frac{4}{5h_s^2}\left(\frac{dw}{dx} - \theta_s\right) \quad ...(14)$$

The above equations are substituted in Equations (1) and (2) which lead to

$$u_c = A_c u_{c0} + B_c u'_c + C_c \theta_c + D_c \phi \quad ...(15)$$

$$u_s = A_s u_{s0} + B_s u'_s + C_s \theta_s + D_s \phi \quad ...(16)$$

where $A_c = 1 - \frac{12y^2}{5h_c^2} + \frac{16y^3}{5h_c^3}$, $B_c = \frac{12y^2}{5h_c^2} - \frac{16y^3}{5h_c^3}$,

$C_c = -y - \frac{4y^2}{5h_c} + \frac{12y^3}{5h_c^2}$, $D_c = \frac{2y^2}{5h_c} + \frac{4y^3}{5h_c^2}$,

$$A_S = 1 - \frac{12y^2}{5h_S^2} - \frac{16y^3}{5h_S^3},\ B_S = \frac{12y^2}{5h_S^2} + \frac{16y^3}{5h_S^3},$$

$$C_S = -y + \frac{4y^2}{5h_S} + \frac{12y^3}{5h_S^2},\ D_S = \frac{2y^2}{5h_S} - \frac{4y^3}{5h_S^2}\ \text{and}$$

$$\varphi = \frac{dw}{dx}.$$

It should be noted that the expressions of u_c and u_s in the above equations contain derivative of w, i.e., dw/dx which demands a C^1 continuous finite element formulation as mentioned earlier. In order to avoid this higher order continuity problem, dw/dx is taken as an independent unknown (ϕ) to have a strain field which will allow a C^0 continuous finite element formulation.

The stress strain relationship at a point of the upper or lower layer of the composite beam may be written as

$$\begin{Bmatrix} \sigma_i \\ \tau_i \end{Bmatrix} = \begin{bmatrix} E_i & 0 \\ 0 & G_i \end{bmatrix} \begin{Bmatrix} \varepsilon_i \\ \gamma_i \end{Bmatrix}$$

or ...(17)

$$\{\bar{\sigma}\}_i = \left[\bar{D}\right]_i \{\bar{\varepsilon}\}_i$$

where σ_i, τ_i, ε_i, γ_i, E_i, and G_i are the normal stress, shear stress, normal strain, shear strain, modulus of elasticity and shear modulus respectively of the i^{th} layer and $i = c$ for the upper layer while $i = s$ for the lower layer. With the help of Equations (15) and (16), the strain vector in the above equation may be expressed as

$$\{\bar{\varepsilon}\}_i = \begin{Bmatrix} \varepsilon_i \\ \gamma_i \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_i}{\partial x} \\ \dfrac{\partial u_i}{\partial y_i} + \dfrac{\partial w}{\partial x} \end{Bmatrix} = [H]_i \{\varepsilon\}_i \quad ...(18)$$

where $[H]_i$ is a function y_i (i.e., dependent on cross-section) whereas $\{\varepsilon\}_i$ is a function of x (i.e., dependent on axial coordinate) and these are as follows.

$$[H]_i = \begin{bmatrix} A_i & B_i & C_i & D_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{dA_i}{dy_i} & \dfrac{dB_i}{dy_i} & \dfrac{dC_i}{dy_i} & \dfrac{dD_i}{dy_i} & 1 \end{bmatrix} \quad ...(19)$$

$$\{\varepsilon\}_i^T = \begin{bmatrix} \dfrac{du_{i0}}{dx} & \dfrac{du_i'}{dx} & \dfrac{d\theta_i}{dx} & \dfrac{d\phi}{dx} & u_{i0} & u_i' & \theta_i & \phi & \dfrac{dw}{dx} \end{bmatrix} \quad ...(20)$$

With the help of Equations (17) and (18), the strain energy of the two layers may be written as

$$U = \frac{1}{2}\int\left(\{\bar{\varepsilon}\}_c^T \{\bar{\sigma}\}_c + \{\bar{\varepsilon}\}_s^T \{\bar{\sigma}\}_s\right)$$

$$dAdx = \frac{1}{2}\int\left(\{\varepsilon\}_c^T [D]_c \{\varepsilon\}_c + \{\varepsilon\}_s^T [D]_s \{\varepsilon\}_s\right)dx \quad ....(21)$$

where $[D]_c = \int\left([H]_c^T \left[\bar{D}\right]_c [H]_c\right)dA_c$ and

$[D]_s = \int\left([H]_s^T \left[\bar{D}\right]_s [H]_s\right)dA_s$.

The cross-section rigidity matrices $[D]_c$ and $[D]_s$ of the two layers presented above are evaluated numerically following the Gauss quadrature integration technique.

The strain energy of the distributed shear springs at the interface between the two layers may be written with the help of Equation (4) as

$$U' = \frac{1}{2}\int k_s s^2 \quad dx = \frac{1}{2}\int k_s \left(u_c' - u_s'\right)^2 dx \quad ...(22)$$

A one dimensional finite element approximation is used to solve the present problem taking u_{co}, u_c', θ_c, ϕ, w, u_{so}, u' and θ_s as the displacement fields. For its implementation, a C^0 continuous isoparametric beam element having three nodes is developed. According to isoparametric formulation (Whitney, 1973), all the field variables are interpolated in terms of their nodal values in an identical manner which may be expressed as

$$\{f\} = \begin{Bmatrix} u_{co} \\ u_c' \\ \theta_c \\ \phi \\ w \\ u_{so} \\ u_s' \\ \theta_s \end{Bmatrix} = \left[N_1[I] \quad N_2[I] \quad N_3[I]\right]$$

$$\begin{Bmatrix} \{\Delta_1\} \\ \{\Delta_2\} \\ \{\Delta_3\} \end{Bmatrix} = [N]\{\Delta\} \quad ...(23)$$

where $\{\Delta_j\} = \{u_{coj}\ u_{cj}'\ \theta_{cj}\ \phi_j\ w_j\ u_{soj}\ u_{sj}'\ \theta_{sj}\}^T$ is a component of the nodal displacement vector $\{\Delta\}$ corresponding to j^{th} node (1, 2 or 3), N_j is the corresponding shape function and $[I]$ is the identity matrix having an order of 8.

The above equation may be substituted in Equation (20) to express the generalised strain vector in terms of nodal displacement vector $\{\Delta\}$ as

$$\{\varepsilon\}_i = \left[[B_1]_i\ [B_2]_i\ [B_3]_i\right] \begin{Bmatrix} \{\Delta_1\} \\ \{\Delta_2\} \\ \{\Delta_3\} \end{Bmatrix} \quad ...(24)$$

$$= [B]_i \{\Delta\}$$

where

$$[B_j]_c = \begin{bmatrix} \frac{dN_j}{dx} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{dN_j}{dx} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{dN_j}{dx} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{dN_j}{dx} & 0 & 0 & 0 \\ N_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_j & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{dN_j}{dx} & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$[B_j]_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{dN_j}{dx} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{dN_j}{dx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{dN_j}{dx} \\ 0 & 0 & 0 & 0 & \frac{dN_j}{dx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_j \\ 0 & 0 & 0 & 0 & N_j & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{dN_j}{dx} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The above equation may be substituted in Equation (21) to express the strain energy of the two layers in terms of their stiffness matrix $[K^l]$ as

$$U = \frac{1}{2}\{\Delta\}^T \left[K^l\right]\{\Delta\} \quad ...(25)$$

where

$$\left[K^l\right] = \int\left([B]_c^T [D]_c [B]_c + [B]_s^T [D]_s [B]_s\right) dx \quad ...(26)$$

Similarly, the strain energy of the interfacial shear springs between the two layers may be expressed in terms of its stiffness matrix $[K']$ with the help of Equation (23) as

$$U' = \frac{1}{2}\{\Delta\}^T [K']\{\Delta\} \quad ...(27)$$

where $[K'] = \int\left([B']^T k_s [B']\right) dx$ ...(28)

In the above equation the matrix $[B']$ may be written as $[B'] = \left[[B'_1]\,[B'_2]\,[B'_3]\right]$ where $\left[B'_j\right] = \left[0\ N_j\ 0\ 0\ 0\ 0\ -N_j\ 0\right]$.

In Equations (15) and (16), *dw*/*dx* is taken as an independent unknown (ϕ) to avoid the higher order inter-elemental continuity problem, but is dependent on *w* and it is simply equal to *dw*/*dx* (i.e., $dw/dx - \phi = 0$) which may cause numerical inconsistencies in some problems. The problem is overcome through satisfaction of the condition $dw/dx - \phi = 0$ variationally using a penulty function approach (Whitney, 1973) taking an addition strain energy as

$$U^p = \int k_p \left(\frac{dw}{dx} - \phi\right)^2 dx \quad ...(29)$$

where k_p is the penalty parameter which will have a large value.

Again this energy can be expressed in terms of its stiffness matrix $[K^p]$ with the help of Equation (23) as

$$U^p = \frac{1}{2}\{\Delta\}^T \left[K^p\right]\{\Delta\} \quad ...(30)$$

where $\left[K^p\right] = \int\left(\left[B^p\right]^T k_p \left[B^p\right]\right) dx$ ...(31)

The matrix $\left[B^p\right]$ in the above equation may be written as $\left[B^p\right] = \left[\left[B_1^p\right]\left[B_2^p\right]\left[B_3^p\right]\right]$ where $\left[B_j^p\right] = \left[0\ N_j\ 0\ 0\ 0\ 0\ -N_j\ 0\right]$.

With the help of Equation (23), the element load vector $\{R\}$ due to a distributed transverse load *q* can be obtained from the work done by the load as

$$W = \int wq \quad dx = \{\Delta\}^T \{R\} \quad ...(32)$$

where $\{R\} = \int\left(\left[N^q\right]^T\right) \quad q \quad dx$ ...(33)

The matrix $\left[N^q\right]$ in the above equation may be written as $\left[N^q\right] = \left[\left[N_1^q\right]\left[N_2^q\right]\left[N_3^q\right]\right]$ where, $\left[N_j^q\right] = \left[0\ 0\ 0\ N_j\ 0\ 0\ 0\ 0\right]$.

The integrations found in Equations (26), (28), (31) and (33) for evaluating the different stiffness matrices and the load vector is carried out numerically following gauss quadrature technique where a full integration rule is applied. In order to avoid any locking problem, the field consistent technique (Reddy, 1984) is use used as mentioned earlier. The stiffness matrix of an element [*K*] can be obtained simply by combing its different components as

$$[K] = [K^l] + [K^r] + [K^p] \quad ...(34)$$

Now the mass matrix and the geometric stiffness matrix of an element can be derived in the similar manner.

With the help of Equation 3 and Equations (15-16), the displacement components at any point within the plate may be expressed as,

$$\{\bar{f}\} = \begin{Bmatrix} u_i \\ w \end{Bmatrix} = [F_i] \begin{Bmatrix} u_{co} \\ u'_c \\ \theta_c \\ \phi \\ w \\ u_{so} \\ u'_s \\ \theta_s \end{Bmatrix} = [F_i][X]\{\Delta\} \quad ...(35)$$

where [F_i] is a matrix of order (2 x 8) which contains A_i, B_i, C_i, D_i and 1. Also [*X*] is a matrix of order (8 x 15), which contains different entries of the corresponding shape functions N_j.

Using the above (Equation 35) the consistent mass matrix of an element can be written as

$$[M] = \int\int\int [X]^T \left([F_i]^T \rho_i [F_i] dz\right)[X] dxdy \quad ...(36)$$

where ρ_i is the mass density of the i^{th} beam component.

In a similar manner the geometric strain vector may be expressed as

$$\{\varepsilon_G\} = \left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u_i}{\partial x}\right)^2\right] = \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial u_i}{\partial x}\right]$$

$$= \frac{1}{2}[A_G][\theta] = \frac{1}{2}[A_G][C_i][N_G][\Delta] \quad ...(37)$$

where [C_i] is a matrix of order (2 x 5) which contains A_i, B_i, C_i, D_i and 1. Also [N_G] is a matrix of order (5 x 24), which contains different entries of the derivatives of corresponding shape functions N_j.

Finally, using the above Equation (37), the geometric stiffness matrix [*G*] of an element can be derived and it may be expressed as

$$[G] = \int\int\int [N_G]^T \left([C_i]^T S_i [C_i] dz\right)[N_G] dxdy \quad ...(38)$$

where the equivalent axial stress of the i^{th} layer, S_i may be expressed (Xu and Wu, 2008) as,

$$S_i = \frac{E_i P}{\sum_{i=1}^{2} E_i h_i}$$ where , E_i and h_i are the modulus of elasticity and depth of the of the *i*-th layer respectively, and *P* is the axial buckling load to be calculated.

Integrations found in the Equations (26), (28), (31), (33), (36) and (38) are carried out numerically following Gauss quadrature integration rule. The element stiffness matrix, geometric stiffness matrix and mass matrix are evaluated for all the elements and assembled together to form the overall stiffness matrix [K_g], mass matrix [M_g] and geometric stiffness matrix [G_g] and these matrices are stored in single array following the skyline storage technique. With these matrices, the governing equation may be expressed as follows

Free vibration: $[K_G]\{\psi\} = \omega^2 [M_G]\{\psi\}$ ...(39)

Buckling: $[K_G]\{\Omega\} = P\lfloor G_g \rfloor\{\Omega\}$ ...(40)

where ω is the frequency of vibration and P is the critical load of buckling. A computer code is written to implement the entire process described above.

RESULTS AND DISCUSSION

In this section studies have been made on different aspects of composite beams using the proposed one dimensional finite element model based on a higher order beam theory (HBT) to assess the performance of the proposed model. A computer code was developed in FORTRAN to implement the proposed model which is used to generate results. Problems of two layered composite beams having rectangular cross-sections (e.g., wooden beam) as well as flanged (Tor I) cross-sections (e.g., concrete wood and steel concrete composite beam) were considered and analyzed for various boundary, material configurations and interlayer shear spring stiffness. As no result is available in the literature on vibration and buckling for the present problem based on HBT, the present results were compared in some cases with the results of the two dimensional analysis by Xu and Wu (2008), TBT and EBT of Xu and Wu (2007) available in the literatures.

Two Layered Concrete-wood Composite Beam

In this example, the problem of free vibration and buckling of a concrete-wood composite beam having T-cross-section is solved to demonstrate the accuracy and convergence of the results obtained by using the proposed one dimensional FE model based on the HBT. The same problem was also solved by Xu and Wu (2007 and 2008) by TBT/EBT [16] and by using the two dimensional analytical method Xu and Wu (2008). The following geometric and material data are used: thickness of the upper layer (concrete) of the beam (h_b) = 50 mm, thickness (depth) of the lower layer (wood) of the beam (h_a) = 150 mm, width of both upper layer beam (b_a) = 300 mm, width of the lower layer beam (= b_b) = 50 mm, modulus of elasticity of the materials(concrete) used in upper layer both the layers, E_a =12 GPa, modulus of elasticity of the materials (wood) used in lower layer both the layers, E_b = 8 GPa, modulus of rigidity of the material used in the upper layer (G_b) = 5 GPa, modulus of rigidity of the material used in the lower layer (G_a) = 3 GPa and the mass densities of the upper and lower layers were considered as 2300 kg/m^3 and 700 kg/m^3 respectively while the interfacial spring stiffness (k_s) values were varied between 0.05 to 500 MPa.

The results of the first six free vibration frequencies (Hz) and first five buckling loads (kN) for a simply supported (SS) beam obtained by the proposed FE model based on HBT were presented in Tables 1 and 3 respectively and were compared with the corresponding two dimensional analytical results by Xu and Wu (2008) and the analytical results based on TBT and EBT Xu and Wu (2007). The beam was analysed in this case by taking interfacial spring stiffness value of 50 MPa. Tables 1 and 3 show that for the conver-gence to the first vibration frequency/ buckling load values, 10 elements (full beam) were needed where as 20 elements were

Table 1: Frequencies of a Simply Supported Concrete-Wood Composite Beam

References	Frequency (Hz)					
	Mode					
	1	2	3	4	5	6
Present (2*)	10.3860	35.5897	124.8049	139.2526	422.9166	528.1592
Present (4)	10.2817	35.5239	68.5368	115.9788	142.7555	218.2233
Present (8)	10.2743	33.1764	65.4658	108.1720	142.2771	162.5441
Present (12)	10.2738	33.1551	65.2701	107.2274	142.2955	159.3763
Present (16)	10.2738	33.1511	65.2355	107.0569	142.2949	158.7934
Present (20)	10.2737	33.1508	65.2190	106.9751	142.2946	158.5104
Present (50)	10.2737	33.1508	65.2189	106.9750	142.2946	158.5103
Xu and Wu (2008) (Error %)	10.2768 (0.03)	33.1771 (0.08)	65.3343 (0.18)	107.3095 (0.31)	–	159.2021 (0.44)
TBT (Xu and Wu, 2007 and 2008) (Error %)	10.3023 (0.28)	33.3569 (0.62)	65.8811 (1.02)	108.6146 (1.53)	–	161.9071 (2.14)
EBT (Xu and Wu, 2007 and 2008) (Error %)	10.3215 (0.47)	33.5264 (1.13)	66.4831 (1.94)	110.1706 (2.99)	–	165.2731 (4.27)

Note: *Indicates no of elements used to model the full beam

Table 2: Frequencies of Concrete-wood Composite Beam for Different Boundary Conditions With Varying Spring Stiffness (k_s)

Boundary Conditions	k_s(MPa)	References Mode	Frequency (Hz)					
			1	2	3	4	5	6
SS	0.05	Present	6.0376	15.5123	23.9907	53.5551	94.2376	142.4958
	0.5	Present	6.1787	24.1339	48.6799	53.6967	94.3762	143.7812
	5	Present	7.2655	25.4623	55.0631	95.7342	124.3589	146.8902
	50	Present	10.2737	33.1508	65.2189	106.9750	142.2946	158.5103
		Xu and Wu (2008)	10.2768 (0.03)*	33.1771 (0.08)	65.3343 (0.18)	107.3095 (0.31)	–	159.2021 (0.44)
	500	Present	11.7472	43.8762	90.2772	142.1714	151.7442	210.5106
SC	0.05	Present	9.3955	30.2249	62.4696	105.5842	142.8872	158.9265
	0.5	Present	9.5014	30.3472	62.5960	105.7109	143.9201	159.0515
	5	Present	10.3905	31.4974	63.8220	106.9540	148.8873	160.2858

Table 2 (Cont.)

Boundary Conditions	k_s(MPa)	References Mode	Frequency (Hz)					
			1	2	3	4	5	6
	50	Present	14.1301	38.9126	73.3273	117.4856	152.8183	171.2583
		Xu and Wu (2008)	14.1376 (0.05)	38.9602 (0.12)	73.4855 (0.22)	117.9755 (0.42)	–	171.9755 (0.42)
	500	Present	17.6803	52.8101	100.7265	153.4819	157.8644	222.3976
CC	0.05	Present	13.5831	37.1140	71.9667	117.4215	172.8277	237.5106
	0.5	Present	13.6616	37.2193	72.6799	117.5374	172.9436	237.6250
	5	Present	14.3881	38.2177	73.1840	118.6766	174.0891	238.7585
	50	Present	18.5664	45.1426	82.0197	128.4705	184.3567	249.1786
		Xu and Wu (2008)	18.5849 (0.1)	45.2227 (0.18)	82.2490 (0.28)	128.9548 (0.38)	185.1945 (0.45)	250.3416 (0.47)
	500	Present	24.6005	61.8541	111.0442	169.1000	234.5998	306.2695
CF	0.05	Present	2.1681	13.44205	37.2989	72.3364	118.1528	142.8872
	0.5	Present	2.3364	13.6241	37.4563	72.5012	118.3088	143.9224
	5	Present	3.1134	15.0876	39.0643	74.0440	119.8090	148.8323
	50	Present	3.9875	19.9772	48.3069	85.2503	131.9393	152.8372
		Xu and Wu (2008)	3.9913 (0.095)	20.0468 (0.35)	48.4782 (0.35)	85.9909 (0.87)	132.2778 (0.26)	–
	500	Present	4.2459	25.0066	64.6993	116.6225	153.4935	176.0746

Note: * Error in percentage shown in parenthesis

required to converge for the remaining frequency/buckling load values. Based on this observation it was decided to carry out all the subsequent analyses with 20 elements in order to get converged solution for the vibration frequencies and buckling loads. In Table 1 and Table 3, the percentage errors of other results with respect to the present results were calculated and were shown in parenthesis. Table 1 and Table 3 show that the present results based on HBT were very accurate as expected as they were very close to those of the two dimensional solution results by Xu and Wu (2008). While there are increasing deviations of the present results from the results based on TBT to EBT, which does not consider any shear deformation into the formulation. These deviations were increased for the vibration frequencies/ buckling loads corresponding to higher modes.

In Tables 2 and 4, the same problem was analysed to study the effect of different

Table 3: Buckling Loads of a Simply Supported Concrete-Wood Composite Beam

References	Frequency (Hz)				
	1	2	3	4	5
Present (2*)	274.068	791.540	3013.843	900.000	900.000
Present (4)	268.547	707.553	1277.456	2087.854	3821.803
Present (8)	268.295	697.973	1202.698	1832.159	2610.606
Present (16)	268.279	697.300	1197.776	1808.793	2533.907
Present (20)	268.278	697.256	1196.791	1806.025	2528.166
Present (50)	268.278	697.256	1196.790	1806.025	2528.165
Xu and Wu (2008) (Error %)	268.6351 (0.13)	699.6742 (0.35)	1205.0415 (0.69)	1826.8929 (1.16)	2570.9989 (1.69)
TBT (Xu and Wu 2007 and 2008) (Error %)	270.0838 (0.67)	708.3848 (1.60)	1229.6902 (2.75)	1883.2759 (4.28)	2683.9653 (6.16)
EBT (Xu and Wu 2007 and 2008) (Error %)	271.0222 (1.02)	714.8772 (2.53)	1249.3871 (4.39)	1929.8718 (6.86)	2779.6112 (9.95)

Note: *Indicates no of elements used to model the full beam

Table 4: Buckling Loads (kN) of Concrete-Wood Composite Beam for Different Boundary Conditions with Varying Spring Stiffness (k_s)

Boundary Conditions	k_s (MPa)	References Mode	Frequency (Hz)				
			1	2	3	4	5
SS	0.05	Present	92.684	365.258	806.858	1400.345	2124.975
	0.5	Present	97.067	369.633	811.130	1404.471	2128.929
	5	Present	134.217	411.445	852.946	1445.211	2168.119
	50	Present	268.278	697.256	1196.790	1806.025	2528.165
		Xu and Wu (2008)	268.6351 (0.13)*	699.6742 (0.35)	1205.0415 (0.69)	1826.8929 (1.16)	2570.9989 (1.69)
	500	Present	350.607	1219.444	2289.942	3377.412	4430.077
SC	0.05	Present	188.263	547.597	1068.041	1730.397	2512.946
	0.5	Present	192.638	551.922	1072.244	1734.441	2516.808
	5	Present	231.632	593.417	1113.361	1774.319	2555.047
	50	Present	436.719	906.677	1463.973	2133.315	2909.313

Table 4 (Cont.)

Boundary Conditions	k_s (MPa)	References Mode	Frequency (Hz)				
			1	2	3	4	5
		Xu and Wu (2008)	437.7220 (0.23)	911.2396 (0.50)	1477.3659 (0.91)	2163.7066 (1.42)	2966.8595 (1.98)
	500	Present	674.799	1689.542	2791.483	3871.008	4912.431
CC	0.05	Present	362.258	734.149	1400.345	2057.110	2958.625
	0.5	Present	369.633	738.432	1404.472	2061.078	2962.388
	5	Present	411.445	779.755	1445.211	2100.220	2999.780
	50	Present	697.256	1101.584	1806.025	2453.514	3350.510
		Xu and Wu (2008)	699.8523 (0.37)	1109.0655 (0.68)	1827.4148 (1.18)	2494.8894 (1.69)	3426.6865 (2.27)
	500	Present	1219.444	2089.881	3377.412	4296.068	5451.758

Note: * Error in percentage shown in parenthesis.

boundary conditions and variation of the spring stiffness (k_s) values; and also to generate new results on the free vibration frequencies and buckling loads respectively for three more boundary conditions (namely, SC: simply supported clamped, CC: Clamped clamped, CF: Clamped free) while the spring stiffness (k_s) values are also varied from 0.05 to 500 Mpa. The present results were compared in Table 2 and Table 4 with the available corresponding results of Xu and Wu (2008) for the spring stiffness value of 50 MPa. The percentage error calculated show the very good accuracy of the present results based on one dimensional HBT while compared with the two dimensional analytical results by Xu and Wu (2008).

Two Layered Wooden Composite Beam Having Rectangular Cross Section

In this example, the problem of the free vibration of a simply supported two-layered rectangular composite beam also studied by Schnabl *et al.* (2006) for static analysis was considered. The beam is having a span of 2.5 m. The problem of free vibration was solved by the proposed finite element (FE) model based on HBT for three different spring stiffness (k_s) values (2.43, 243 and 24300 MPa). For the composite beam, the following geometric and material data were used. For the composite beam, the following geometric and material data are used: Thickness of the upper layer of the beam (h_b) = 200 mm, thickness of the lower layer of the beam (h_a) = 300 mm, width of both the layers of the beam ($b_a = b_b$) = 300 mm, modulus of elasticity of the materials used in both the layers = $E_a = E_b$ = 12000 MPa, modulus of rigidity of the material used in the upper layer (G_b) = 800 MPa, modulus of rigidity of the material used in the lower layer (G_a) = 1200 MPa. The mass density for both the upper and lower component was considered as 700 kg/m^3.

The present results for the first six vibration frequencies were presented in Table 5. All these results obtained by the proposed model based on HBT are new and found to follow the expected trend.

A Continuous Steel Concrete Composite Beam Having Flanged Cross-Section

The problem of a two span continuous beam having an overall length (L) of 6.706 m was studied in this example. The beam is roller supported at the right end, pinned at the left end and supported on an intermediate roller which divides the beam into two equal spans of 3.353 m. The cross section (Figure 2) consists of a rectangular concrete slab (482.6 mm x 60.325 mm) and a steel joist having 76.2 mm x 9.58 mm flanges and a 135.25 mm x 9.58 mm web. The modulus of elasticity for the steel joist was taken as 206964 MPa while that of concrete was 27594.3 MPa. The shear modulus of the joist was 82785.6 MPa and that of concrete was 11497.6 MPa. The mass densities of the upper (concrete) and lower

Table 5 Frequencies of a Simply Supported Wooden Composite Beam

Spring Stiffness (MPa)	Frequency (Hz)					
	Mode					
	1	2	3	4	5	6
2.43	31.0952	74.7896	256.2310	399.5388	483.4809	725.6948
243	93.3301	213.8044	275.2490	498.3639	568.3352	737.2565
24300	120.7400	290.2644	359.8142	595.9671	839.0874	937.4604

Figure 2: Typical Cross Section of A Flanged Composite Beam

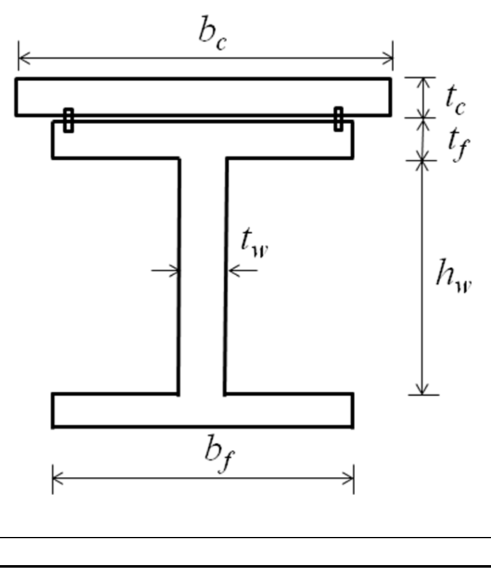


Figure 3: Frequencies (Hz) of a Continuous Steel Concrete Composite Beam For Different Values of Spring Stiffness

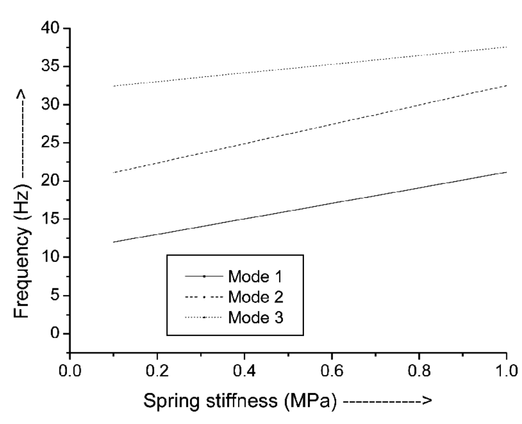
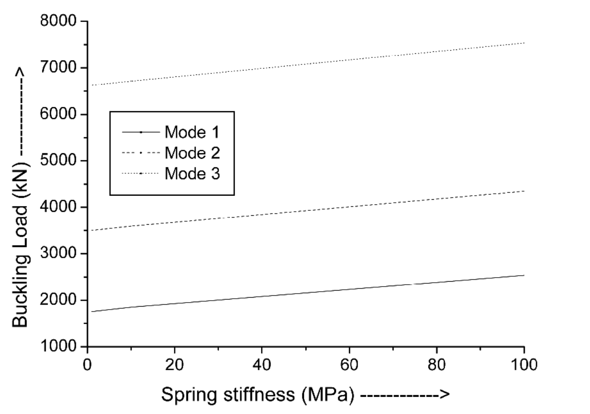


Figure 4: Buckling Load (kN) of a Continuous Steel Concrete Composite Beam for Different Values of Spring Stiffness



layer (steel) were considered as 2400 kg/m^3 and 7850 kg/m^3 respectively. The variation of frequencies of vibration (Hz) and buckling loads (kN) with different values of the interfacial spring stiffness obtained by using the proposed FE model based on HBT are plotted in Figures 3 and 4 respectively which again show an expected trend of the results.

CONCLUSION

A new finite element model based on a higher order beam theory was proposed for the free vibration and buckling analysis of composite beams having two material layers with partial shear interaction between these two layers. The partial shear interaction was modeled by introducing distributed linear shear springs at the interface of the two beam components. A third order variation of the axial displacement over the beam depth was adopted in the proposed model which helps to get the actual parabolic variation of shear stress. The proposed model also satisfies shear stress free conditions at the top and bottom surfaces of the beam. It helps to eliminate the need of an arbitrary shear correction factor dependent on cross-sectional geometry as required in TBT. A three node C^0 continuous isoparametric beam finite element based on a displacement approach is developed for the implementation of the higher order beam theory. The finite element formulation was made field consistent and a full numerical integration of the stiffness matrix was carried out in order to avoid any shear locking and stress oscillation problem as well as to improve the solution accuracy. The proposed finite element model was validated by using it to solve numerical examples of composite beams and the results obtained were compared with the published results. The numerical analysis shows the applicability of the proposed FE model based on HBT in predicting vibration frequencies and buckling loads of composite beams with partial interaction closer to the exact solutions and more accurately than the existing models based on EBT and TBT. Some new results were presented, which should be useful for future research, as there was no published result on composite beams based on HBT.

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