Lyapunov Stability for Four-Story Buildings using the Fundamental Equations of Constrained Motion

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Abstract—This paper is concerned with the stability problem of four-story buildings. It is one of important issues for building safety. The Lyapnov stability theory is applied to four-story buildings, which are dynamical systems represented by mass-spring-damper model. The main objective is to analyze control effort to stabilize the mass-spring-damper model of four-story buildings by using the Fundamental Equations of Constrained Motion and Lyapunov theory. In the initial stage, mass-spring-damper models are oscillated freely without any conditions. Then, we want to stabilize the models using Lyapunov theory by adding additional control forces to control motions of four-story buildings. This paper shows the effectiveness of an application of Lyapunov stability theory with the Fundamental Equation of Constrained Motion to four-story buildings.

Index Terms—Mass-Spring-Damper model, four-story building, the Fundamental Equation, Constraint Equation, Stability, Lyapunov theory

I. INTRODUCTION

Stability of a story-building is an important topic to focus. Wind load and earthquake are examples of huge impacts to structure of a story-building, which are the main causes of fatigue or structural collapse. Many methods to solve those problems have thus been studied for the past years. A tuned-mass-damper (TMD) is a well-known machinery for reducing vibration of structures, which is installed at the top of a tall building. It consists of additional mass, spring and damper. Dynamic equations of a tall building subjected to earthquake are determined by Hamilton’s principle and finite element method [1]. But sometime the TMD system can be installed at the base of structure [2]. The effectiveness of a TMD depends on the mass, damper and frequency of the TMD. In addition, Rüdinger F. [3] studies influence of tuned mass dampers with nonlinear viscous damping elements. There are many other methods that are similar to using TMD system such as; hybrid mass liquid damper (HMLD) [4], which is the system of a rigid water tank, that is attached an additional mass to the main structure through a spring and damper system for vibration control. A Zener model is a parallel combination of a spring and an elastically supported damper, which can hold up from free and forced vibration [5]. In another worthy method [6-7], friction dissipaters are applied at every floor for seismic protection of building structures. Using friction dampers to designing and controlling for two buildings, seismic responses can be reduced [8]. Moreover, Semi-active friction-type multiple tuned mass dampers (SAF-MTMD) is more advantage and requires less installation space than a passive friction-type multiple tuned mass dampers (PF-MTMD) [9]. SAF-MTMD can be used for a low-intensity earthquake. Proportional-Derivative (PD), Proportional-Integral-Derivative (PID) controllers [10] and partial floor loads [11] are other options to use for vibration control. Partial floor loads are used as multiple TMDs. Some other papers [12-13] have developed and analyzed mass-spring-damper system to estimate the deformation of free damped and forced damped systems in various viscous fluids. In [14-16], Lyponov theorem has been studied to stability the dynamic of story-buildings.

In this work, a new method to analyze a four-story building, which uses the Fundamental Equations of Constrained Motion is presented. It is a new aspect of equations of motion which described explicit general equations of motion for dynamical system [17]. The constraints of these systems could be holonomic or non-holonomic. The equations of motion are the simplest and the most general to use [18]. For example, Udadia F.E., and Phohomsiri P. (2006) [19] demonstrate a new general explicit form of the equations of motion for constrained systems with singular mass matrices and applied to multi-body systems. This paper shows the solving of basic dynamic systems by adding more generalize coordinates than needed to created equation of motion. Next, Udadia F.E., and Wanichanon T. (2013) [20] explicate a new simple general explicit form of the equations of motion for general constrained systems. The constraints could be holonomic or non-holonomic, and the mass matrices could be positive semi-definite or positive definite. Furthermore, they [21] applied the Fundamental Equation of Constrained Motion to a multi-body system.
in 2014, which was a triple pendulum. The energy of the first mass to be zero was the constraint of the system.

From the previous researches, the Fundamental Equation of Constrained Motion (FECM) was a new viewpoint. The FECM could be used with all dynamic systems. It is a closed-form equation, which can use the same constraint to control the system. It has three main parts: i) unconstrained motion, ii) constraint equation and iii) constrained motion.

II. FUNDAMENTAL EQUATION OF CONSTRAINED MOTION

We use the knowledge of the Fundamental Equation of Motion [21] for a mass-spring-damper system model of a four-story building.

Firstly, we make the unconstrained system in which its coordinates are all assumed independent of each other. The general equation of the unconstrained motion is

\[ M(q,t) = Q(q, \dot{q}, t) \]  

with the initial conditions

\[ q(t = 0) = q_0, \dot{q}(t = 0) = \dot{q}_0. \]  

The acceleration of the uncontrolled system is given by multiplying equation (2.1) with \( M^{-1} \)

\[ \ddot{q} = M^{-1}(q, \dot{q}, t)Q(q, \dot{q}, t) = a. \]  

In (2.1), \( q \) and \( \dot{q} \) are displacement and velocity of the system respectively, \( \ddot{q} = a \) is acceleration of the uncontrolled system, \( M \) is the mass matrix, \( Q \) is the generalized force vector and \( t \) is the time.

Second step, we define constraints for the uncontrolled system. The constraints can be functioning dependent. The function of constraint is given by

\[ \varphi_i(q, \dot{q}, t) = 0, \quad i = 1, 2, ..., n, \]  

where \( n \) is number of requirement constraints. The constraints in (2.4) can be linear or non-linear which include the varieties of holonomic and/or nonholonomic constraints.

Differentiating the constraint equation (2.4) with respect to time \( t \), we get

\[ A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t), \]  

where \( A \) is a constraint matrix and \( b \) is a constraint vector. The rows of \( A \) and \( b \) have \( n \) dimension, which is the number of requirement constraints in equation (2.4).

For the third step, we use the information from the two steps before. That is the unconstrained motion (2.1) and the constraint equation (2.5), then combining it. Thus the Fundamental Equation of Constrained Motion is

\[ M(q, t)\ddot{q} = Q(q, \dot{q}, t) + Q^c(q, \dot{q}, t), \]  

where \( Q^c \) is the control force that makes sure the control requirements in Eq. (2.5) are assured. Using \( M \) and \( Q \) from (2.1) and \( A \) and \( b \) from (2.5), we get the control force \( Q^c \) by

\[ Q^c(q, \dot{q}, t) = A^T(M^{-1}A^T)^+(b - Aa). \]  

In (2.7), \( a = M^{-1}Q \) is the acceleration of the uncontrolled system. Representing the control force from Eq. (2.7) in Eq. (2.6), the general equation of constrained system is given by

\[ M(q, t)\ddot{q} = Q + A^+(AM^{-1}A^T)^+(b - Aa), \]  

where the superscript \( \text{“}^+\text{”} \) denotes the Moore-Penrose (MP) inverse of a matrix [21].

Pre-multiplying both side of equation (2.8) with \( M^{-1} \), we obtain the acceleration of the constrained system as

\[ \ddot{q} = M^{-1}Q + M^{-1}A^T(AM^{-1}A^T)^+(b - Aa). \]  

After explaining the use of the Fundamental Equation of Constrained Motion. We further explain the dynamic of a four-story building represented by mass-spring-damper system that is make use of the FECM.

III. DYNAMIC OF FOUR-STORY BUILDING

In real buildings, all of components of the structure interact to resist applied loads. When the building objected to an earthquake, it can be idealized into a simple one-dimensional model for building vibration. In this paper, a four-story building as shown in Fig. 1 is studied.

![Figure 1. Diagram of the four-story building.](image-url)

Consider the response of dynamic of the four-story building subjected to an earthquake motion as shown in Fig. 2.

![Figure 2. Free body diagram of the four-story building.](image-url)
where:

\[ m_i \text{ represents dead weight and live weight of each floor. } k_i \text{ is a spring constant, } c_i \text{ is a damping coefficient, } F_i \text{ is an external force (such as a wind motion). } x_i \text{ is a displacement. } \ddot{x}_g \text{ represents as a ground acceleration.} \]

Each floor is represented by its mass and the supporting equipment is idealized by a spring constant representing resistance to lateral motion and a damping coefficient providing frictional energy losses.

From the Newton’s second law of motion

\[ \sum F = m \ddot{x}. \quad (3.1) \]

The equations of motion of the four-story building as shown in Fig. 2. are

1st floor:

\[ m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 + (c_1 + c_2) \dot{x}_1 = 0 \]

2nd floor:

\[ -c_2 \ddot{x}_2 = F_1 - m_1 \ddot{x}_g \]

3rd floor:

\[ m_3 \ddot{x}_3 + (k_3 + k_4) x_3 - k_3 x_2 - k_4 x_4 + (c_3 + c_4) \dot{x}_3 = 0 \]

4th floor:

\[ m_4 \ddot{x}_4 - k_4 x_3 + k_4 x_4 - c_4 \dot{x}_3 + c_4 \dot{x}_4 = F_4 - m_4 \ddot{x}_g \]

Then, we group Eqs (3.2)-(3.5) in the matrix form [13]

\[ M \ddot{x} + C \dot{x} + K x = F - M \ddot{x}_g \]

(3.6)

where:

\[ M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \quad r = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \]

\[ C = \begin{bmatrix} (c_1 + c_2) & -c_2 & 0 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 & 0 \\ 0 & -c_3 & (c_3 + c_4) & -c_4 \\ 0 & 0 & -c_4 & c_4 \end{bmatrix} \]

\[ K = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 & 0 \\ 0 & -k_3 & (k_3 + k_4) & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \]

Using the first step of FECM (Eq. (2.1)), Eq. (3.7) is rearranged as

\[ M \ddot{x} + C \dot{x} + K x = F - M \ddot{x}_g \quad \text{as:} \]

\[ M \ddot{x} + C \dot{x} + K x = F - M \ddot{x}_g \quad \text{Similarly, we get the uncontrolled acceleration} \]

\[ a = M^{-1}(F - C \dot{x} - K x) \quad \text{as:} \]

\[ V = a^T e + a_2 e^T \dot{e} + 2a_2 \dot{e}^T e \quad \text{as:} \]

\[ V = -\alpha V \quad \text{where} \]

\[ \dot{V} = -\alpha V \quad \text{Differentiating the Lyapunov candidate (3.11), we get} \]

\[ \dot{V} = 2a_1 \dot{e}^T \dot{e} + 2a_2 \dot{e}^T \dot{e} + 2a_2 \dot{e}^T e \quad \text{and set the constraint equation} \]

\[ V = -\alpha V \quad \text{Differentiating the Lyapunov candidate (3.11), we get} \]

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\[ A = \begin{bmatrix} \sum_{i=1}^{n} (2\eta_i \dot{q}_i + 2\eta_i \ddot{q}_i) & \sum_{i=1}^{n} (2\eta_i \dot{q}_i + 2\eta_i \ddot{q}_i) & \sum_{i=1}^{n} (2\eta_i \dot{q}_i + 2\eta_i \ddot{q}_i) & \sum_{i=1}^{n} (2\eta_i \dot{q}_i + 2\eta_i \ddot{q}_i) \\ -2\eta_1 (\dot{q}_1 + \dot{x}_1 \ddot{q}_1 + \ddot{x}_1^2 \dddot{q}_1 + \dddot{x}_1^3 \ddddot{q}_1) & -2\eta_2 (\dot{q}_2 + \dot{x}_2 \ddot{q}_2 + \ddot{x}_2^2 \dddot{q}_2 + \dddot{x}_2^3 \ddddot{q}_2) & -2\eta_3 (\dot{q}_3 + \dot{x}_3 \ddot{q}_3 + \ddot{x}_3^2 \dddot{q}_3 + \dddot{x}_3^3 \ddddot{q}_3) & -2\eta_4 (\dot{q}_4 + \dot{x}_4 \ddot{q}_4 + \ddot{x}_4^2 \dddot{q}_4 + \dddot{x}_4^3 \ddddot{q}_4) \\ \end{bmatrix} \]

\[ b = \begin{bmatrix} \eta_1 (\dddot{x} \dddot{q}_1 + \dddot{x}_2 \dddot{q}_2 + \dddot{x}_3 \dddot{q}_3 + \dddot{x}_4 \dddot{q}_4) \\ a_1 (\dddot{x}_1 + \dddot{x}_2 \dddot{q}_1 + \dddot{x}_3 \dddot{q}_2 + \dddot{x}_4 \dddot{q}_3) \\ -a_2 (\dddot{x}_1 + \dddot{x}_2 \dddot{q}_2 + \dddot{x}_3 \dddot{q}_3 + \dddot{x}_4 \dddot{q}_4) \\ -2a_2 (\dddot{x}_1 + \dddot{x}_2 \dddot{q}_2 + \dddot{x}_3 \dddot{q}_3 + \dddot{x}_4 \dddot{q}_4) \end{bmatrix} \]

(3.17)

The final step, we use the information of equations (3.8) and (3.16) into the equation (2.9), we will get the constrained acceleration \( (\dddot{q}) \). And we can find other kinematical information such as velocity \( (\dot{q}) \) and displacement \( (q) \) by appropriately integrating the acceleration \( (\dddot{q}) \).

IV. NUMERICAL SIMULATION

In this section, the proposed approach to stabilize the four-story building is verified by a numerical solution using Matlab. We start by defining the properties of four-story building model as

- Mass of each floor: \( m_i = 5,740.39 \text{ kg} \)
- Damping of each floor: \( c_i = 13,121.77 \text{ N} \cdot \text{s/m} \)
- Spring stiffness of each floor: \( k_i = 2,999.471 \text{ kN/m} \)
- External force: \( F_i = 12.491 \text{ kN} \)
- Ground acceleration: \( \dddot{x}_g = \sin 30^\circ \cdot g \)

Time: \( t = 100 \text{ s} \)

The following mass, damping coefficient, spring stiffness and external force are tested from [6].

Next, we assign the initial displacement of the building as

\[ \begin{bmatrix} x_1, x_2, x_3, x_4 \end{bmatrix}^T = \begin{bmatrix} 1.2830, 1.2830, -5.1112, -5.1112 \end{bmatrix}^T \]

and initial velocity as

\[ \begin{bmatrix} \dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4 \end{bmatrix}^T = \begin{bmatrix} 9.0886, 9.0886 \end{bmatrix}^T. \]

Using the FECM (Eq. (2.9)), the displacement responses of the four-story building model are obtained as shown in Fig. 3 - Fig. 6.

The displacement of the first floor in Fig. 3 decreases closer to zero before 10 seconds. Fig. 4 shows the displacement of the second floor which reduces from 1.2830 m to 0 m in approximately the same time with the first floor. Fig. 5 and Fig. 6 show the displacement of the third and fourth floors which drop from -5.1112 m to 0 m in the same direction.

V. CONCLUSION

This paper presents the application of the Fundamental Equation of Constrained Motion with the Lyapunov stability theorem to the problem of stabilization of the four-story building. The control forces to stabilize the four-story building, which is represented by the mass-spring-damper system, is numerically computed. The results show that the proposed control forces can reduce displacement errors and satisfactorily stabilize each floor of the four-story building.

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REFERENCES


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