A One-Dimension Kinematic Hardening Model Based on Continuous Hyperplasticity

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***Abstract*—This paper presents a one-dimension kinematic hardening model based on continuous hyperplasticity with infinite number of yield surface. Continuous hyperplasticity is a development of hyperplasticity theory, an approach to plasticity theory based on thermodynamics principles. It gives ability to develop many sophisticated engineering models that can describe more realistic behavior. In order to apply to numerical analysis, the discretization from infinite number of yield surface to multiple-yield-surface is shown. Applications to 1-D Finite element model using rate-dependent solution will be mentioned in this paper. The results show that this is a promising theory that can be describe nonlinear elasto-plastic response of material. By a suitable choice of some parameters, realistic behavior of a model can be derive.**

***Index Terms*—multiple-yield-surface, rate-dependent, cyclic loading, kinematic hardening, thermodynamic, elasto-plastic.**

I. INTRODUCTION

Hyperplasticity theory is an approach to plasticity theory based on thermodynamics principle which has the root from [1] and developed to a rigorous and consistent frame work for a wide range of engineering material by [2]- [4]. This concept emphasizes on the use of internal variable and the formulation from the first and second law of thermodynamic ensures that the response of model will automatically obey these laws. One of the advantages of this concept is that entire behavior of a model can be derive by only two potential function.

Continuous hyperplasticity, which is introduced in [5], is the next step in the development of hyperplasticity concept. From this work, the use of internal variable is extended to the internal function which gives ability to derive models with infinite number of yield surface. Potential functions become potential functionals, which should be known as function of function.

Other development of hyperplasticity theory, which can be found in [6], is the development for rate-dependent materials. From this advanced, the rate-dependent solution is derived to get over the numerical obstacle happening in multiple-yield-surface model.

Many researches have applied the hyperplasticity theory for engineering model, see [7]- [10]... However, little has been done, such as [7] and [9], by employed the continuous hyperplasticity.

This paper will focus on the one-dimension model. The discretization from infinite number of yield surface to multiple-yield-surface is introduced in case of rate-dependent solution. Applications and the results show the advantages of this concept and effect of some parameters will be discussed.

II. ONE-DIMESION MODEL BASED ON CONTINUOUS HYPERPLASTICITY

A fundamental of kinematic hardening formulation can be found in [11], this section introduces briefly the functionals in a slight different form.

Firstly, the Gibbs free energy functional is presented, which takes the form:

$$\hat{g}(\sigma,\alpha) = -\frac{1}{2E}\sigma^2 + \int_0^1 \frac{H}{2}\alpha^2 d\eta - \sigma\int_0^1 \hat{\alpha} d\eta \qquad (1)$$

where $\hat{\alpha}$ and $\hat{H}$ are the internal function and the kinematic hardening function, respectively. The hat notation expresses the variables are functions of a dimensionless parameter η, which varies from 0 to 1.

The generalized stress is define as:

$$\hat{\bar{\chi}} = -\frac{\partial \hat{g}}{\partial \hat{\alpha}} = \sigma - \hat{H}\hat{\alpha} \qquad (2)$$

For rate-independent materials, the second functional is the yield functional, which in one dimension model usually takes the form:

$$\hat{y} = |\hat{\chi}| - \hat{k} \qquad (3)$$

In which $\hat{\chi}$ is the dissipative generalized stress and it should be noted that $\hat{\bar{\chi}} = \hat{\chi}$ is the hypothesis of the hyperplasticity concept, which can be found in [2]- [3].

The flow potential functional is used instead of yield functional in case of rate-dependent solution.

As [6], flow potential functional can be defined as:

$$\hat{w} = \frac{\langle \hat{y} \rangle^2}{2\mu} = \frac{\langle |\hat{\chi}| - \hat{k} \rangle^2}{2\mu} \qquad (4)$$

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where μ is viscosity factor and $\langle\ \rangle$ is the Macaulay bracket which operates as $\langle x\rangle = x$ if $x > 0$ and $\langle x\rangle = 0$ if $x \le 0$.

The plastic strain rate is as below:

$$\dot{\hat{\alpha}} = \frac{\partial \hat{w}}{\partial \hat{\chi}} = \frac{\left\langle |\chi| - \hat{k} \right\rangle S(\chi)}{\mu} \tag{5}$$

where $S(\ )$ is generalized signum function that $S(x) = 1$ if $x > 0$, $S(x) = -1$ if $x < 0$ and $S(x)$ undefined if $x = 0$

The incremental stress-strain response using rate-dependent behavior is written as:

$$\dot{\varepsilon} = \frac{1}{E}\dot{\sigma} + \int_0^1 \dot{\hat{\alpha}}\, d\eta \tag{6}$$

Eq. (6) can be convert to the incremental form as:

$$\delta\varepsilon = \frac{1}{E}\delta\sigma + dt\int_0^1 \frac{\left\langle |\chi| - \hat{k} \right\rangle S(\chi)}{\mu} d\eta \tag{7}$$

where dt is the time increment that shown the effect of time step in rate-dependent solution.

III. The Discretization from Continuous Model to Multiple-Yield-Surface Model

The formulation in previous section is written in continuous functions. In order to apply to numerical analysis, it is necessary to establish a discretization formulation. The model with infinite number of yield surface is now defined in case of multiple-yield- surface model.

The mechanical behavior of one-dimensional kinematic hardening with multiple-yield-surface model is expected to be as shown in Fig. 1. Each yield surface active whenever **σ** get over slip value k_i. The hardening parameter H_i can be seen as the modulus of a spring when yield occur and the viscous elements represent the rate effect.

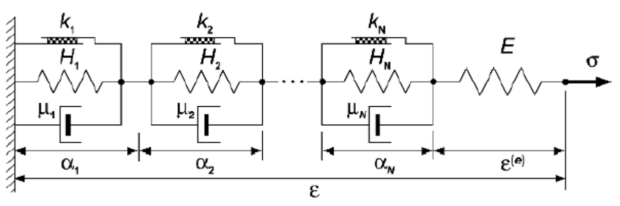

Figure 1. Mechanical behavior of multiple-yield-surface model.

The Gibbs free energy functional in (1) can be rewritten in form:

$$g = -\frac{1}{2E}\sigma^2 + \sum_{i=1}^{N} \frac{H_i \alpha_i^2}{2} d\eta_i - \sigma \sum_{i=1}^{N} \alpha_i d\eta_i \tag{8}$$

where N is the number of yield surface and $\eta_i = \frac{i}{N}$ $(i = 1...N)$. Thus, $d\eta_i = \frac{i}{N} - \frac{i-1}{N} = \frac{1}{N}$

Therefore, (8) can be rewritten as:

$$g = -\frac{1}{2E}\sigma^2 + \frac{1}{N}\sum_{i=1}^{N} \frac{H_i \alpha_i^2}{2} - \frac{\sigma}{N}\sum_{i=1}^{N} \alpha_i \tag{9}$$

The hat notations are abandoned to show that the variables are no longer functions, they become a series of discretized values.

The generalized stress corresponding to α_i is now defined as:

$$\overline{\chi}_i = -N\frac{\partial g}{\partial \alpha_i} = \sigma - H\alpha_i \tag{10}$$

As [7], Hardening function should takes the form:

$$H_i = A.B.(C - \eta_i)^n \tag{11}$$

where *A, B, C, n* are parameters for defining the hardening values.
In this research, the Hardening function is chosen as:

$$H_i = \frac{E(1-\eta_i)^2}{2} \tag{12}$$

The yield function takes the form:

$$y_i = |\chi_i| - k_i \tag{13}$$

In order to control initial size, k_i is defined as:

$$k_i = k_0.\eta_{0i} = k_0\left(\eta_0^{initial} + \left(1 - \eta_0^{initial}\right)\frac{i}{N}\right) \tag{14}$$

where k_0 is the value of outer most yield surface and the $\eta_0^{initial}$ is the parameter to control the value of inner yield surfaces ($0 \le \eta_0^{initial} \le 1$)

Thus, the flow potential takes the form:

$$w_i = \frac{\langle y_i\rangle^2}{2\mu} = \frac{\left\langle |\chi_i| - k_i \right\rangle^2}{2\mu} \tag{15}$$

And the plastic strain rate:

$$\dot{\alpha}_i = \frac{\partial w_i}{\partial \chi_i} = \frac{\left\langle |\chi_i| - k_i \right\rangle S(\chi_i)}{\mu} \tag{16}$$

Finally, the incremental response:

$$\dot{\varepsilon} = \frac{1}{E}\dot{\sigma} + \frac{1}{N}\sum_{i}^{N} \dot{\alpha}_i \tag{17}$$

Or we can rewrite in other form:

$$\delta\varepsilon = \frac{1}{E}\delta\sigma + dt\frac{1}{N}\sum_{i}^{N} \frac{\left\langle |\chi_i| - k_i \right\rangle S(\chi_i)}{\mu_i} \tag{18}$$

The rate dependent solution give ability to get over problem when using multiple-yield-surface model. For each load increment it is assumed that the active yield surface will translate to the stress point after a period of time (*dt*) to ensure the consistent condition of yield surfaces.

IV. Implemented To The 1-D Finite Element Model

Model is tested with a 1-D finite element as shown in Fig. 2, a bar fixed at one end and loaded at the other. Numerical example is implemented to Matlab program and Newton-Raphson method is used for the nonlinear analysis.

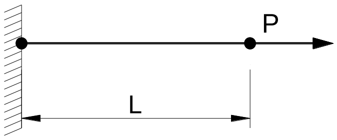

Figure 2. 1-D Finite element model.

Material properties are chosen based on the properties of Steel, Young modulus *E=210000Mpa*, Yield strength σ_P*=240Mpa*.

Geometric parameter: Length of element *L=100mm*, Section area $A=200mm^2$.

Loading P is varied in order to view the advantages more convenient.

V. Numerical Example And Discussion

The results focus on the effect of number of yield surface, $\eta_0^{initial}$ parameter and the ability of hyperplasticity models to capture behaviors under cyclic loading. For the influences of other parameters, there should be further researches and experimental data.

A. Effect of Nuber of Yield Surface

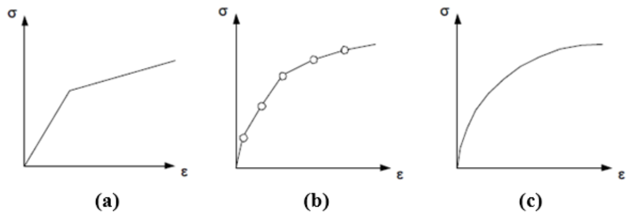

Figure 3. The stress-strain responses in cases of a) bilinear, b) piecewise linear and c) nonlinear

In the conventional approach to elasto-plastic behavior using single-yield-surface model, the stress-strain response is bilinear. The elastic moduli controls the stiffness within yield surface and hardening moduli affects when stress point touch the yield surface. By using multiple-yield-surface concept, the stress-strain response becomes piecewise linear and when the number of yield surface increase to infinite number, the stress-strain behavior is similar to nonlinear behavior as shown in Fig. 3.

In the first example, the loading process P is increase from *0* to *54kN*. The value of other parameter: $\eta_0^{initial} = 0$ and number of yield surface *(N)* is varied.

The first result is shown in Fig. 4. The 1 yield surface model show bilinear behavior. For the case *N=2*, the yield strength of first and second yield surface calculated from (14) are *120MPa* and *240MPa*, respectively. Therefore, it can be seen from Fig. 4 that whenever the stress get over yield strength of a yield surface, the yield surface is active and slope of stress-strain behavior change. If number of yield surface increase, the behavior is piecewise linear as the result of 6 yield surfaces model.

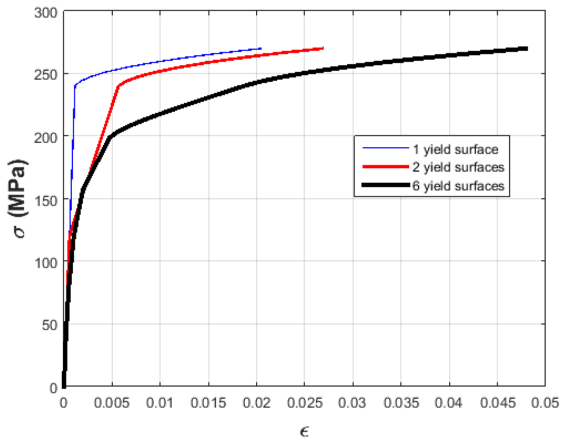

Figure 4. The stress-strain behaviors for cases when number of yield surface *N = 1, 2* and *6*.

Consider the case when *N=10* and 20, results of these models are shown in Fig. 5.

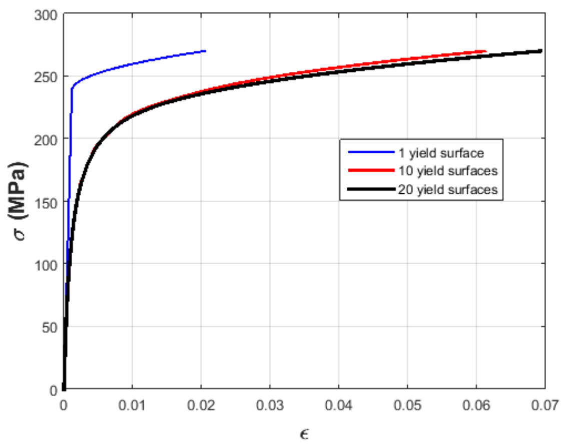

Figure 5. The stress-strain behaviors for cases when number of yield surface N=1, 10 and 20.

It can be seen from Fig. 5 that for the model of 10 and 20 yield surfaces, the behaviors are similar to the nonlinear as shown in Fig. 3. In addition, the difference of stress-strain response between these models are not too much. Therefore, by a suitable choice in number of yield surface, the nonlinear behavior of model can be derived.

B. Influence of Initial Value $\eta_0^{initial}$

The meaning of value $\eta_0^{initial}$ is to control the yield strengths of each yield surfaces. For multiple-yield-surface model, the outer most yield surface is coincident with the yield surface of single-yield-surface model. The distribution of inner yield surfaces is uniform and depend on $\eta_0^{initial}$. Fig. 6 shows an example of the distribution of yield strength in case of *N=5*, $\eta_0^{initial}$ *=0* and *0.5*.

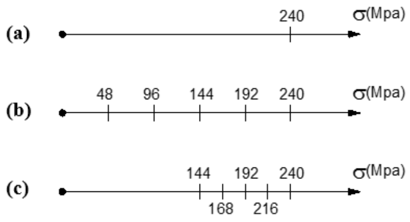

Figure 6. Distribution of yield strength of yield surfaces in case of a) 1 yield surface, b) 5 yield surfaces and $\eta_0^{initial}=0$ and c) 5 yield surfaces and $\eta_0^{initial}=0.5$

In the second example, the values of P is similar to first example (from *0* to *54kN*), number of yield surface is *N=10* and the value of $\eta_0^{initial}$ is varied.

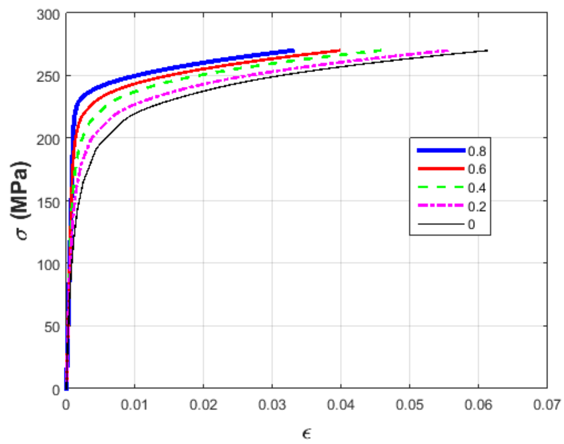

Figure 7. The effect of $\eta_0^{initial}$ to the stress-strain responses.

The result in Fig. 7 show the influence of $\eta_0^{initial}$ to the stress-strain response. It is clear that the smaller initial value used, the earlier plastic behavior occurs. Thus, by appropriate choice of initial value, the behavior of model can be calibrated.

C. Behavior under Cyclic Loading

One of the key advanced of hyperplasticity theory is the ability to capture behavior of models under cyclic loading. The continuous hyperplasticity and the discretization to multiple-yield-surface give more advantages to describe the nonlinear relationship especially in unloading process. In this section, the behavior of single-yield-surface model and multiple-yield-surface model (with 10 yield surfaces) is consider.

The initial value $\eta_0^{initial}=0$. Two loading processes are used as show in Table I.

TABLE I. LOADING PROCESS (MPA)

Step	Process	
	1	2
1	From 0 to 45	From 0 to 54
2	From 45 to -46.5	From 54 to -54.75
3	From -46.5 to 48	From -54.75 to 57

The behaviors of two models when applying two loading processes are shown below.

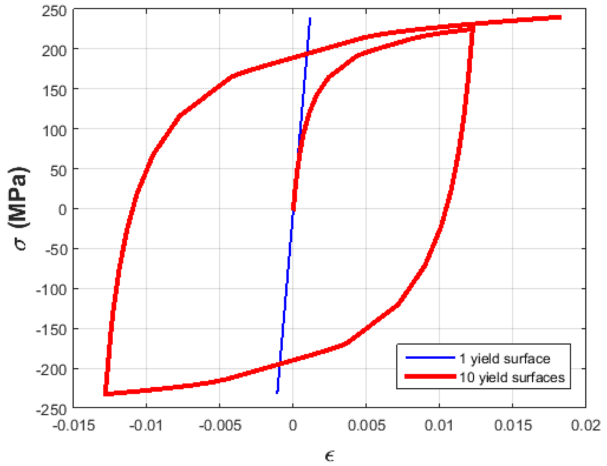

Figure 8. Stress-strain responses for Process 1.

In the first loading process, the load cause the stress that not get over the yield strength in case of single-yield-surface model, thus the behavior of this model is purely elastic. For multiple-yield-surface model, the plastic behavior occurs very early, so the stress-strain response is nonlinear as shown in Fig. 8.

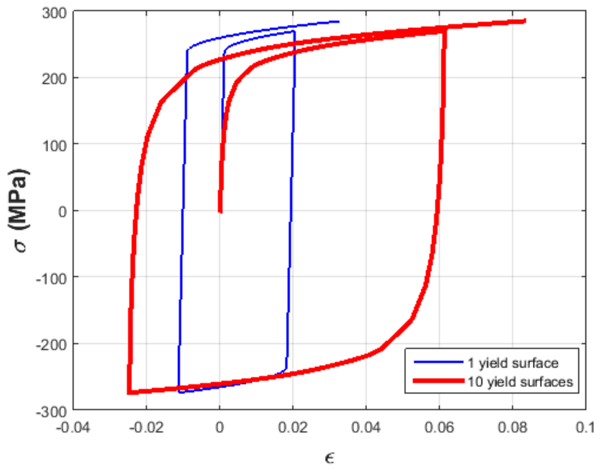

Figure 9. Stress-strain responses for Process 2

In the second loading process, when yield occurs in single-yield-surface model and all the yield surfaces are active in the multiple-yield-surface model, the responses are shown in Fig. 9. The advantages of multiple-yield-surface model is clearly seen. In many kind of material, there is no area with purely elastic behavior. For these material, the plastic response appears very early and the single-yield-surface model cannot capture the realistic nonlinear behavior. By employing the multiple- yield-surface model, these problem can be easily gotten over.

If the loading-unloading processes continue to apply on multiple- yield-surface model, the result is as shown in Fig. 10.

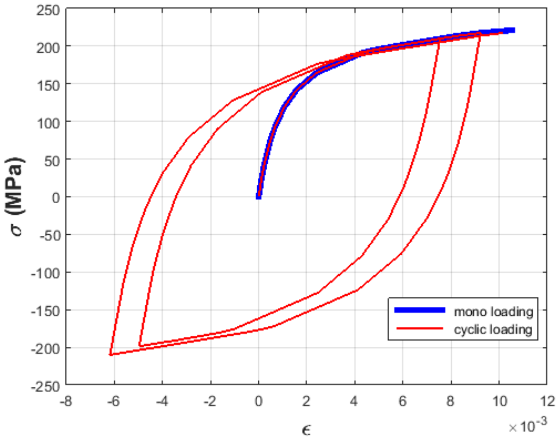

Figure 10. Behaviors of multiple-yield-surface model under mono loading and cyclic loading.

The ability for capturing behavior of hyperplasticity model under cyclic loading can be clearly seen in Fig. 10. The characteristic of kinematic hardening is also shown. If the cyclic loading continue, behavior will follow the Masing rule as discuss in [7], which describe the relationship between response of cyclic loading and mono loading processes as shown in Fig. 10 and can be briefly stated:

- The unloading and reloading curves should follow the initial loading curve (backbone curve) if the previous maximum strain is exceeded.
- If the current loading or unloading curve intersects the curve described by previous loading or unloading curve, the stress-strain relationship follows the previous curve.

VI. Conclusion

The results show the advantages of the continuous hyperplasticity theory. By a suitable choice of parameters, the theoretical model can describe a realistic nonlinear behavior of material, especially for those with cannot be found a purely elastic response. For a particular material model, there should be further researches which calibrate the results of theoretical model and data from experience.

Ability to capture behavior of model under cyclic loading is also shown in this paper. This is an advanced of plasticity theory based on thermodynamics principles. The results of application to 1-D Finite element models give the possibility for the extension to multi-dimension models, which could be done in further research.

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