Nonlinear Bending Analysis of Isotropic Plates Supported on Winkler Foundation Using Element Free Galerkin Method

Gaurav Watts, M. K. Singha, and S. Pradyumna
Department of Applied Mechanics, Indian Institute of Technology Delhi, New Delhi, India
Email: gauravwatts.bits@gmail.com, {maloy, pradyum}@am.iitd.ac.in

***Abstract*—This paper deals with the nonlinear bending analysis of rectangular plates resting on Winkler type foundation using Element Free Galerkin (EFG) method with Moving Kriging (MK) shape function. The nonlinear equations of equilibrium, based on first order shear deformation theory and von Kármán's strain-displacement relationship are solved using Newton-Raphson method. Limited parametric study is conducted to examine the effectiveness of the present method in solving elastically supported plates with different loading and boundary conditions.**

***Index Terms*—plate on winkler foundation, element free galerkin method, moving kriging shape function**

I. Introduction

Most of the civil engineering structures are built on foundation slabs (called raft slab), which rests directly on the elastic soil. Hence, plates / slabs on continuous elastic foundation have received considerable attention of the researchers in the last few decades. Winkler model is the simplest soil-structure interaction model in which the deflection is assumed to be dependent only on the contact pressure. Both analytical and numerical approaches were used in the past to investigate plates resting on elastic foundation. Voyiadjis and Kattan [1] presented analytical solutions for thick rectangular plates on Winkler foundation using Navier and Levy type methods. Kobayashi and Sonoda [2] provided the Levy type solutions for plates with two sides simply supported and resting on Winkler foundation. Svec [3] investigated the same problem using finite element method with the triangular mesh. Liew *et al*. [4] presented numerical solutions for a similar problem using differential quadrature method. Ng and Chan [5] carried out geometrically nonlinear analysis of clamped plates of different shapes resting on Winkler foundation using collocation least square technique. Existing literature on plate resting on elastic foundation contains various solutions for linear and nonlinear problems which are not included here for brevity.

Meshless methods, due to their ease of adaptivity have an edge over mesh based techniques in handling large deformation problems. One of the most popular meshfree methods, called Element Free Galerkin (EFG) method [6]-[8] based on Moving Least Squares (MLS) shape function, received much attention in solving linear and nonlinear problems with complicated loading and boundary conditions. Since MLS shape functions do not satisfy Kronecker delta property, effective imposition of boundary conditions remained a topic of interest for years [9]. To eliminate this problem, Gu [10] used along with EFG, a new shape function based on geostatistical technique called Moving Kriging (MK) method. Various applications of EFG based on Moving Kriging (MK) can be found in Ref [11]-[13].

In the present work, element free Galerkin method with MK based shape functions is employed for linear and nonlinear bending analysis of isotropic thin and moderately thick rectangular plates resting on Winkler foundation subjected to patch and line loads. Meshfree formulation based on First order Shear Deformation Theory (FSDT) and von Kármán's strain-displacement relationship is solved using Newton-Raphson method. The effect of different loading, boundary conditions and subgrade modulus (K) on the nonlinear bending behaviour is studied. Results obtained are compared and are found to be in good agreement with the available analytical and numerical solutions the literature.

II. Moving Kriging Based Shape Function

The approximation of a field variable using MK based interpolating shape function is given by [10]:

$$u^h(x) = [p^T(x)A + r^T(x)B]u = \sum_{I=1}^{n} \phi_I(x) u_I \qquad (1)$$

where $p(x)=\{1\ x\ y\ x^2\ y^2\ xy\}$ is a quadratic polynomial basis . A, B and rare given by :

$$A = (P^T R^{-1} P)^{-1} P^T R^{-1}, \quad B = R^{-1}(I - PA) \qquad (2)$$

$$r^T(x) = \{cor(s_1, x)\ \ \ldots.\ \ldots\ \ \ldots.\ \ cor(s_n, x)\} \qquad (3)$$

A is an $m \times n$ matrix, **B** is an $n \times n$ matrix while **I** is an $n \times n$ unit matrix, where m denotes the number of terms in the polynomial basis and n denotes the number of nodes whose influence domain contains the point of

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interest, x. Size of influence domain, d_m is given by αd_c where d_c is the average nodal spacing and α is the non-dimensional scaling factor. In the present study, shape of influence domain for all the nodes are taken as rectangular while value of α is taken as 3.0.

r is a 1 × n vector of correlation between the given nodes (s) and point of interest x. $cor(x_i, x_j)$ is a correlation function. In the present formulation, Gaussian correlation function is used which is given by :

$$cor(x_i, x_j) = e^{-\theta r_{ij}^2} \tag{4}$$

where, $r_{ij} = \dfrac{h_{ij}}{d}, h_{ij} = \left\| x_i - x_j \right\|$. d is the maximum distance up to which the correlation between the nodes exist. In the present formulation, coordinates are normalized so that the values of x_i and y_i lie between 0 and 1 while d is taken as unity. ***P*** and ***R*** in (2) are given by:

$$P = \begin{bmatrix} p_1(s_1) & \dots & p_m(s_2) \\ \dots & \dots & \dots \\ p_1(s_n) & \dots & p_m(s_n) \end{bmatrix}$$

$$R = \begin{bmatrix} cor(s_1, s_1) & \dots & cor(s_1, s_n) \\ \dots & \dots & \dots \\ cor(s_n, s_1) & \dots & cor(s_n, s_n) \end{bmatrix} \tag{5}$$

III. Geometrical Nonlinear Analysis

First order shear deformation theory with five degrees of freedom $(u, v, w, \theta_x, \theta_y)$ per node is employed here to model the isotropic plate resting on elastic foundation. The nonlinear algebraic equations of equilibrium for the plate and the linearized version of the same may be written as:

$$[K_L + K_{NL}(\delta)]\delta = F - K_w \delta \tag{6}$$

$$K_T \Delta\delta = \Delta F \;\; K_T \Delta\delta = \Delta F \tag{7}$$

where K_L and K_{NL} are linear and nonlinear stiffness matrices respectively; K_T is the tangent stiffness matrix; K_w is consistent foundation stiffness matrix; F is the load vector and K_σ is the geometric stiffness matrix. The stiffness matrices may be expressed as:

$$K_L = \iint_\Omega \begin{bmatrix} \psi_m^T A \psi_m & \psi_m^T B \psi_b \\ \psi_b^T B \psi_m & \psi_b^T D \psi_b + B_\gamma^T A_s B_\gamma \end{bmatrix} dxdy$$

$$K_{NL} = \iint_\Omega \begin{bmatrix} 0 & \dfrac{\psi_m^T A B_L}{2} \\ B_L^T A \psi_m & \dfrac{\psi_b^T B B_L}{2} + B_L^T B \psi_b + \dfrac{B_L^T A B_L}{2} \end{bmatrix} dxdy$$

$$K_{NLT} = \iint_\Omega \begin{bmatrix} 0 & \psi_m^T A B_L \\ B_L^T A \psi_m & \psi_b^T B B_L + B_L^T B \psi_b + B_L^T A B_L \end{bmatrix} dxdy$$

$$K_\sigma = \iint_\Omega G^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} G \, dxdy,$$

$$K_w = \iint_\Omega \psi_w{}^T k_w \psi_w \, dxdy$$

$$K_T = K_L + K_W + K_{NLT} + K_\sigma \tag{8}$$

Here, k_w is the elastic subgrade modulus. The dimensionless elastic subgrade modulus, K used in the present study is given by:

a) $K = (k_w a^4/D)^{1/4}$ for all linear problems;

b) $K = k_w a^4/D$ for all nonlinear problems

Equation (6) is solved by Newton-Raphson method to trace the nonlinear bending behavior of elastically supported rectangular plates.

IV. Numerical Results

The efficiency of element free Galerkin's method with moving kriging shape function for the nonlinear bending analysis of elastically supported plates is studied here. Rectangular isotropic plates with length "a", width "b" and thickness "h" is assumed to be supported on the Winkler foundation at the bottom. The edges of the plate may be free (FFFF) or simply supported (SSSS) or clamped (CCCC) and the corresponding boundary conditions may be expressed as

SSSS: $v = w = \theta_y = 0$ at $x = 0, a$

while $u = w = \theta_x = 0$ at $y = 0, b$

CCCC: $u = v = w = \theta_x = \theta_y = 0$ at all edges.

FFFF: No boundary condition at the edges

At the beginning, accuracy of the present method and the in-house computer code for bending analysis of isotropic rectangular plates resting on the *Winkler foundation* is examined by studying the following two example problems for which results are available in the literature:

TABLE I. Convergence of Maximum Deflection from the Linear Bending Analysis of Square Plate Resting on Winkler Foundation Subjected to Patch Load with SSSS Boundary Condition.

a/b = 1, a/h = 100, K = 3, ν = 0.3, θ = 20, $\bar{w} = 10^3 w_{max} D/q_0 a^4$			
Nodes	**$\bar{W}$ [14]**	**$\bar{W}$ - Present**	**Error(in %)**
Patch Load - u/a = v/b = 0.5			
10 x 10	1.775	1.727	2.672
12 x 12		1.762	0.713
14 x 14		1.769	0.323
16 x 16		1.769	0.325
18 x 18		1.773	0.111

Example 1: Isotropic square plate resting on Winkler foundation with simply supported or clamped edges.

Convergence of maximum deflection of simply supported (SSSS) isotropic square plates resting on Winkler foundation and carrying a square patch load (Fig. 1) or line load (Fig. 2) is presented in Table I and Table II respectively. It is observed that 18 × 18 nodes are sufficient to obtain converged results for both patch load and line load and the maximum deflection match well with the results of Ref [14] obtained using improved differential quadrature method.

TABLE II. CONVERGENCE OF MAXIMUM DEFLECTION FROM THE LINEAR BENDING ANALYSIS OF SQUARE PLATE RESTING ON WINKLER FOUNDATION SUBJECTED TO LINE LOAD WITH SSSS BOUNDARY CONDITION.

a/b = 1, a/h = 100, K = 3, ν = 0.3, θ = 20, $\overline{w} = 10^3 w_{max} D/Q_0 a^3$			
Nodes	$\overline{W}$ **[14]**	$\overline{W}$ **- Present**	**Error(in %)**
Line Load - x_L = 0.5a, v/b = 0.5			
10 x 10	4.128	3.990	3.361
12 x 12		4.086	1.035
14 x 14		4.106	0.553
16 x 16		4.109	0.464
18 x 18		4.121	0.169

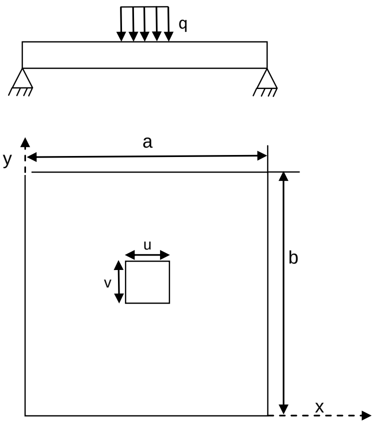

Figure 1. Isotropic rectangular plate subjected to patch load

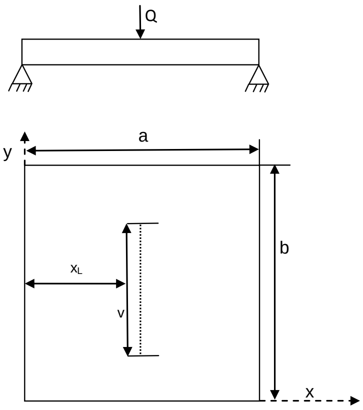

Figure 2. Isotropic rectangular plate subjected to line load

Further, the central displacement of the plate subjected to both patch load and line load are found to match well with the results of Ref [14] in Table III and Table IV respectively for different boundary conditions and dimensionless subgrade modulus (K).

TABLE III. COMPARISON OF MAXIMUM DEFLECTION OF SQUARE PLATES RESTING ON WINKLER FOUNDATION SUBJECTED TO PATCH LOAD (U/A = 0.5, V/B = 0.5) WITH DIFFERENT BOUNDARY CONDITIONS AT THE EDGES.

a/b = 1, ν = 0.3, θ = 20, Nodes = 18 × 18, $\overline{w} = 10^3 w_{max} D/q_0 a^4$			
a/h	$\overline{W}$ **[14]**	$\overline{W}$ **- Present**	**Error(in %)**
SSSS - K = 3			
5	2.1300	2.1348	0.2255
100	1.7746	1.7726	0.1112
SSSS - K = 5			
5	0.9460	0.9458	0.0139
100	0.8477	0.8464	0.1487
CCCC - K = 3			
5	1.2652	1.2683	0.2417
100	0.8003	0.7936	0.8468

TABLE IV. COMPARISON OF MAXIMUM DEFLECTION OF SQUARE PLATES RESTING ON WINKLER FOUNDATION SUBJECTED TO LINE LOAD (X_l= 0.5a, V/B = 0.5) WITH DIFFERENT BOUNDARY CONDITIONS AT THE EDGES.

a/b = 1, ν = 0.3, θ = 20, Nodes = 18 × 18, $\overline{w} = 10^3 w_{max} D/Q_0 a^3$			
a/h	$\overline{W}$ **[14]**	$\overline{W}$ **- Present**	**Error(in %)**
SSSS: K = 3			
5	5.3705	5.3736	0.0580
100	4.1285	4.1214	0.1710
SSSS: K = 5			
5	2.7056	2.6967	0.3285
100	2.0624	2.0570	0.2576
CCCC: K = 3			
5	3.5153	3.5148	0.0140
100	2.0333	2.0174	0.7792

Example 2: *Isotropic square plate resting only on Winkler foundation (edges are free)*

In order to confirm the validity of the present formulation, a comparison of results is made for a rectangular plate with free edges, resting on Winkler foundation and subjected to loading conditions as shown in Fig. 3. Due to symmetric boundary conditions, only lower quarter of plate is considered. It can be observed from Table V and Fig. 4 that results obtained using the present formulation are in very good agreement with those provided by Choi and Kim [15]. Further, 18 × 18 nodes are sufficient to obtain converged results for multiple path loads.

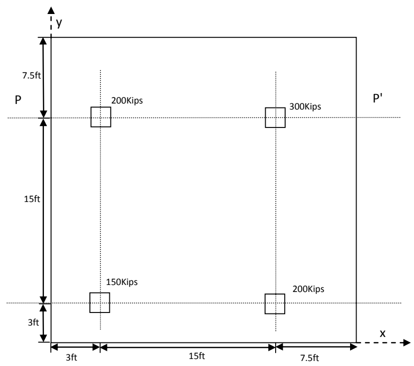

Figure 3. Rectangular plate subjected to patch load [15]

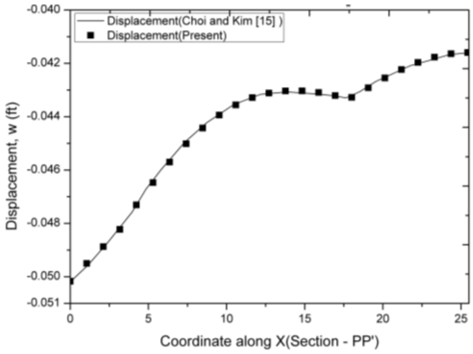

Figure 4. Comparison of linear static deflection for plate on Winkler foundation subjected to patch loads [15]

TABLE V. CONVERGENCE STUDY FOR LINEAR STATIC DEFLECTION OF PLATE RESTING ON WINKLER FOUNDATION SUBJECTED TO PATCH LOADS [15]

a = b = 51ft, h = 2ft E = 3122ksi (Concrete), ν = 0.15, k_w = 28.4k/ft^3 (4467kN/m^3)			
Nodes	**w_{max}(ft) [15]**	**w_{max}(Present)**	**Error(in %)**
6 x 6	0.05945	0.05852	1.563
8 x 8		0.05913	0.532
10 x 10		0.05925	0.328
12 x 12		0.05928	0.278
14 x 14		0.05931	0.237

It is observed that the present method is quite efficient for the bending analysis of plates with different loading and boundary conditions. Now, additional problems of square isotropic plates supported only on Winkler foundation (FFFF: edges are free) are taken up for investigation.

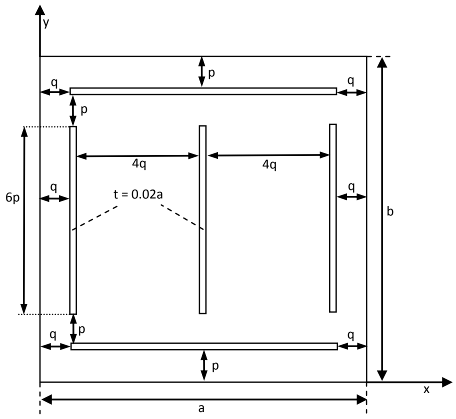

Figure 5. Geometry and loading conditions (Case - I) for square plate resting on Winkler foundation with free edges

Problem 1: Square plate resting on elastic foundation at the bottom and supporting multiple patch loads from walls.

A square plate (a/h = 50) with free edges resting on Winkler foundation at the bottom and subjected to multiple patch loads, as shown in Fig. 5 is considered to simulate the behavior of foundation subjected to load transferred by walls in a structure. It should be noted here that all the patch loads considered are of equal thickness (t = 0.02a). All patch loads are assumed to be uniformly distributed over the patch area and are of equal magnitude (q_0). Nodal distribution and background mesh used in the present case is shown in Fig. 6. Non-dimensional deflections ($\overline{w} = wD/q_0a^4$, ν =0.3) for different subgrade modulus of soil (K) and first loading case (Case - I) are presented in the form of contour diagrams shown in Fig. 7(a) - Fig 7(c). Maximum non-dimensional deflection for low subgrade modulus (K= 4) is 2.9343e-04, which reduces to 2.2839e-05 for moderate subgrade modulus (K= 8) and further go down to 1.9948e-06 for very high value of soil modulus (K= 16) .

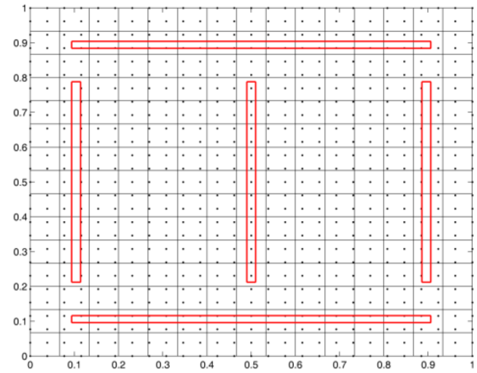

Figure 6. Nodal distribution and background mesh for square plate resting on Winkler foundation with free edges (Case - I)

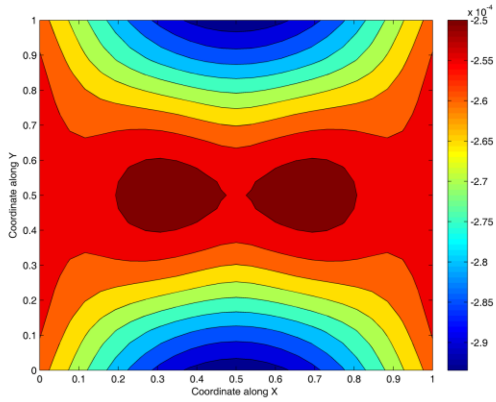

Figure 7. (a) Non-dimensional deflection - Case - I (K = 4, ν =0.3) ($\overline{w}_{max}$ = 2.9343e-4)

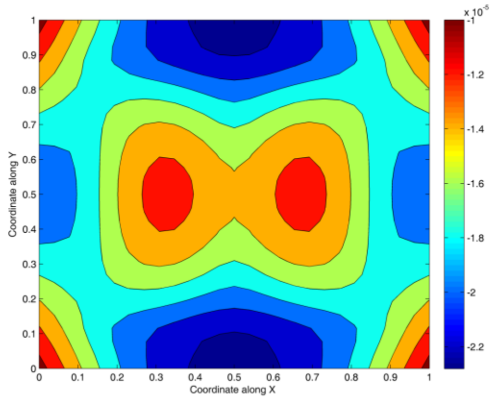

Figure 7. (b) Non-dimensional deflection - Case - I (K = 8, ν =0.3) ($\overline{w}_{max}$ = 2.2839e-5)

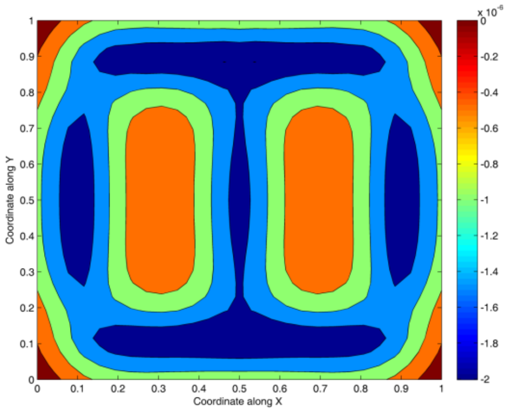

Figure 7. (c) Non-dimensional deflection -Case - I (K = 16, ν =0.3) ($\bar{w}_{max}$ = 1.9948e-06)

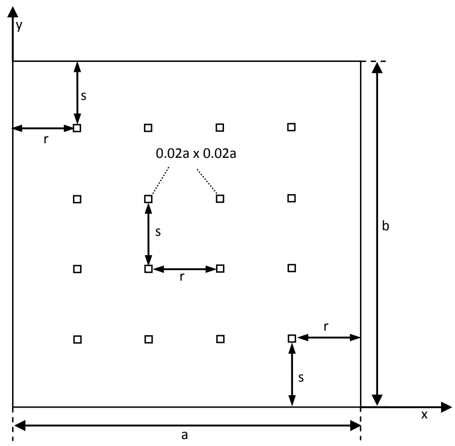

Figure 8. Geometry and loading conditions (Case - II) for square plate resting on Winkler foundation with free edges (r = s)

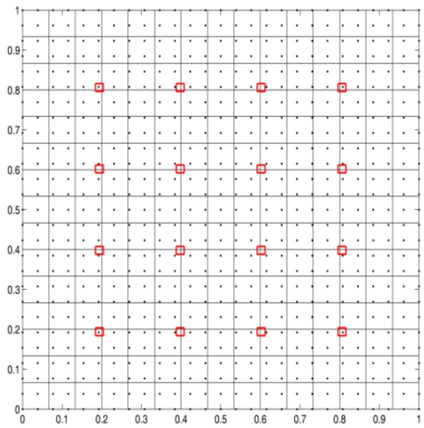

Figure 9. Nodal distribution and background mesh for square plate resting on Winkler foundation with free edges (Case - II)

Problem 2: Square plate resting on elastic foundation at the bottom and supporting multiple patch loads from columns.

In this case, deflection due to load transferred by columns on the raft slab of a structure is modeled by considering a square plate (a/h = 50) resting on elastic foundation subjected to various patch loads as shown in Fig. 8. All patch loads are of equal size (0.02*a* × 0.02*a*) and are uniformly spaced in the domain (*r* = *s*). Each patch load is assumed to be uniformly distributed over the patch area. Uniform distribution of nodes and rectangular background mesh employed for numerical integration is shown in Fig. 9. Contour diagrams for non-dimensional static deflection ($\bar{w} = wD/q_0 a^4$, ν =0.3) are given by Fig 10(a) - Fig. 10(c). For *K* = 4, maximum deflection of the plate is 3.2000e-05. It reduces to 2.7062e-06 with increase in value of subgrade modulus to 8. With further increase in value of nondimensional modulus to 16, defection reduces to 2.3303e-07.

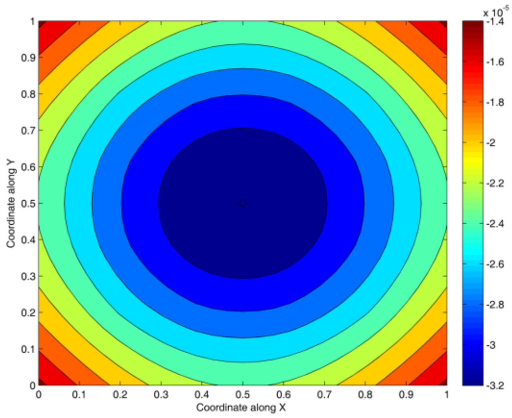

Figure 10. (a) Non-dimensional deflection - Case- II (K = 4, ν =0.3) ($\bar{w}_{max}$ = 3.2000e-05)

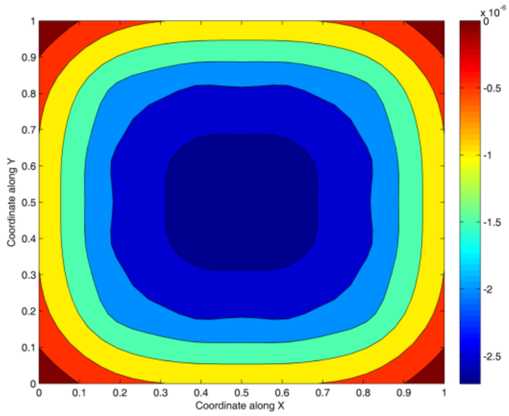

Figure 10. (b) Non-dimensional deflection - Case- II (K = 8, ν =0.3) ($\bar{w}_{max}$ = 2.7062e-06)

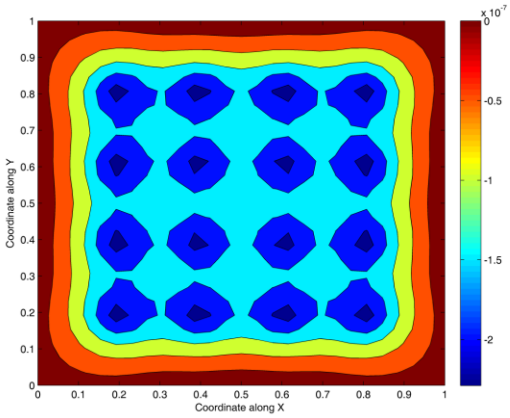

Figure 10. (c) Non-dimensional deflection- Case - II (K = 16, ν =0.3) ($\bar{w}_{max}$ = 2.3303e-07)

It is quite apparent from the contour plots that as the values of subgrade modulus of soil increases, the deflection of the plate decreases due to the increased stiffness of the soil. It can also be noted from the Fig 7. and Fig. 10 that the plate deforms uniformly for small values of subgrade modulus whereas for higher value of soil modulus, plate deformation is local and is only near to the load application area.

Problem 3: Nonlinear bending analysis of isotropic rectangular plate resting on Winkler foundation with different subgrade modulus (K) subjected to patch load or line load with clamped edges.

Load-deflection curves for geometrically nonlinear bending analysis of clamped square plate subjected to patch load and line load for different subgrade modulus are presented in Fig. 11 and Fig. 12 respectively. A comparison of result is also made for nonlinear bending analysis of thin plate resting on Winkler foundation subjected to uniformly distributed load (UDL) and is found to match well in Fig. 11 with Ref [5].

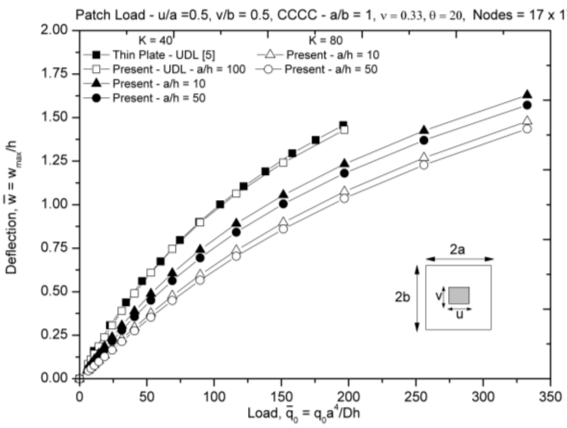

Figure 11. Non-dimensional load versus deflection curve for nonlinear bending analysis of plate resting on Winkler foundation subjected to Patch Load .

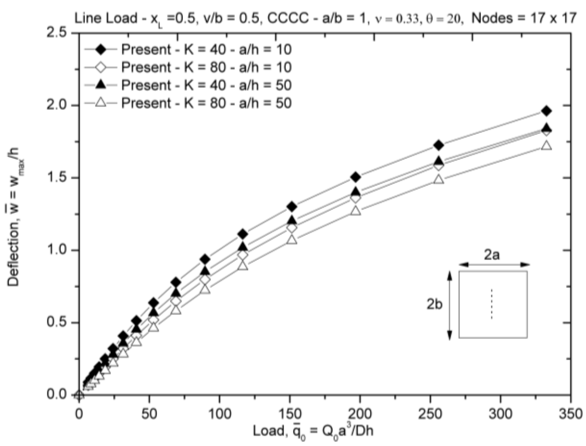

Figure 12. Non-dimensional load versus deflection curve for nonlinear bending analysis of plate resting on Winkler foundation subjected to line load.

It is observed that the non-dimensional deflection reduces as the non-dimensional subgrade modulus increases from 40 to 80. Further, for a particular non-dimensional load parameter, the non-dimensional deflection for the thicker plate (*a/h* = 10) is more compared to thinner plate (*a/h* = 50) due the effect of shear deformation.

V. Conclusions

Element free Galerkin method based on first order shear deformation theory and MK shape function is employed to investigate linear and nonlinear bending behavior of isotropic thin and thick rectangular plates resting on Winkler foundation. Numerical studies include effect of different loading conditions, boundary conditions and subgrade modulus on maximum deflection. It is observed from the results that the method is accurate and reliable for solving complex nonlinear problems. From the limited parametric study, it is observed that, the deformation of the plate and compression of the elastic foundation (settlement of foundation) is more localized near the loaded region for higher values of subgrade modulus and thicker plate. The compression of foundation becomes more uniform for lower values of subgrade modulus.

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Mr. Gaurav Watts is a research scholar in the Department of Applied Mechanics, Indian Institute of Technology Delhi. He obtained his Master's degree from Department of Mechanical Engineering, Birla Institute of Technology and Science, Pilani, Rajasthan. He completed his undergraduate studies from Department of Mechanical Engineering at Thapar University, Patiala, Punjab. His research interest is meshfree methods in linear and nonlinear problems of solid mechanics.

Dr. M K Singha obtained his Ph.D degree from the Department of Civil Engineering, Indian Institute of Technology Kharagpur in 2002. Dr. Singha is currently working as an Associate Professor in the Department of Applied Mechanics, IIT Delhi. He has supervised three Ph.D students and co-authored more than 30 research papers in the area of composite and FGM panels, finite element method and impact mechanics.

Dr. S. Pradyumna received his Ph.D degree from the Department of Civil Engineering, Indian Institute of Technology Kharagpur in 2009. His areas of interest include dynamics and stability of plates and shells, finite element method, meshless methods, composite structures and functionally graded materials. Dr. Pradyumna is working as an Assistant Professor in the Department of Applied Mechanics, Indian Institute of Technology Delhi since 2010. He has seven years of teaching experience and has published/presented more than 35 papers in peer reviewed international journals and conferences.